DERIVATIONS OF APPLIED MATHEMATICS

VOLUME TWO OF TWO, INCLUDING PART III

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Part III

Transforms and special functions

Chapter 17

The Fourier series

It might be said that, among advanced mathematical techniques, none is so useful, and few so appealing, as the one Lord Kelvin has acclaimed "a great mathematical poem."[1] It is the Fourier transform, which this chapter and the two that follow it will develop. This first of the three chapters brings the Fourier transform in its primitive guise as the *Fourier series.*

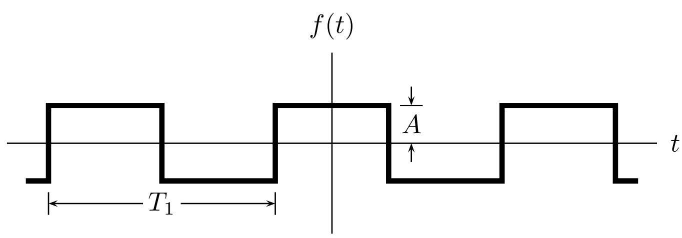
The Fourier series is an analog of the Taylor series of chapter 8 but meant for *repeating waveforms,* functions $f(t)$ of which

$$f(t) = f(t + nT_1),\ \Im(T_1) = 0,\ T_1 > 0,\ \text{for all } n \in \mathbb{Z}, \tag{17.1}$$

where T_1 is the waveform's characteristic *period.* Examples include the *square wave* of Fig. 17.1. A Fourier series expands such a repeating wave-

[1][89, chapter 17]

Figure 17.1: A square wave.



form as a superposition of complex exponentials (or, equivalently, if the waveform is real, as a superposition of sinusoids).

Suppose that you wanted to approximate the square wave of Fig. 17.1 by a single sinusoid. You might try the sinusoid at the top of Fig. 17.2—which is not very convincing, maybe, but if you added to the sinusoid another, suitably scaled sinusoid of thrice the frequency then you would obtain the somewhat better fitting curve in the figure's middle. The curve at the figure's bottom would yet result after you had added in four more sinusoids respectively of five, seven, nine and eleven times the primary frequency. Algebraically,

$$\begin{aligned} f(t) \quad = \quad & \frac{8A}{2\pi}\bigg[\cos\frac{(2\pi)t}{T_1} - \frac{1}{3}\cos\frac{3(2\pi)t}{T_1} \\ & \qquad + \frac{1}{5}\cos\frac{5(2\pi)t}{T_1} - \frac{1}{7}\cos\frac{7(2\pi)t}{T_1} + \cdots\bigg]. \end{aligned} \tag{17.2}$$

How faithfully (17.2) really represents the repeating waveform and why its coefficients happen to be $1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \ldots$ are among the questions this chapter will try to answer; but, visually at least, it looks as though superimposing sinusoids worked.

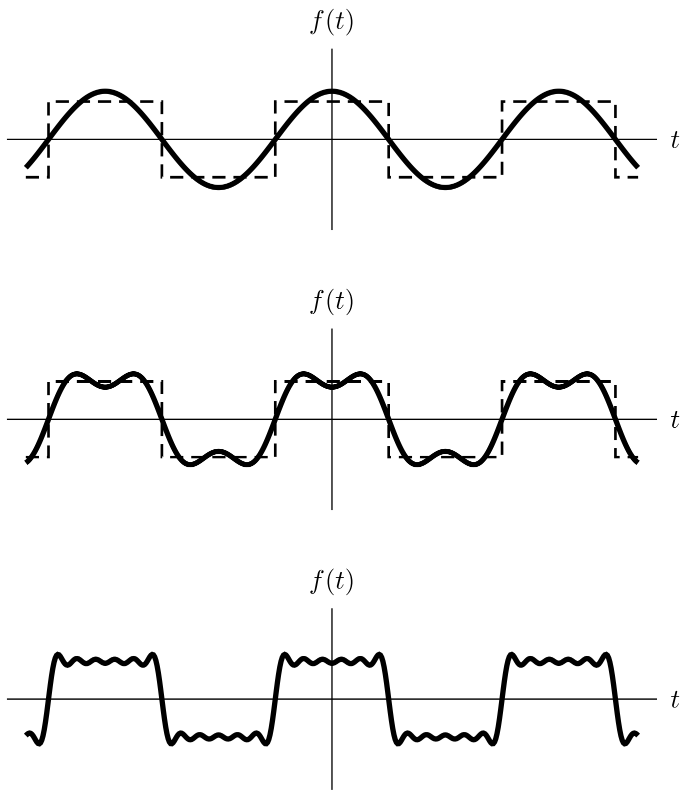
The chapter begins in preliminaries, starting with a discussion of Parseval's principle.

17.1 Parseval's principle

Parseval's principle is that *a step in every direction is no step at all.* In the Argand plane (Fig. 2.7), stipulated that

$$\begin{aligned} \Delta\omega\, T_1 &= 2\pi, \\ \Im(\Delta\omega) &= 0, \\ \Im(t_o) &= 0, \\ \Im(T_1) &= 0, \\ T_1 &\neq 0, \end{aligned} \tag{17.3}$$

Figure 17.2: Superpositions of one, two and six sinusoids to approximate the square wave of Fig. 17.1.



and also that[2]

$$\begin{aligned} j, n, N &\in \mathbb{Z}, \\ n &\neq 0, \\ |n| &< N, \\ 2 &\leq N, \end{aligned} \tag{17.4}$$

the principle is expressed algebraically as that[3]

$$\int_{t_o - T_1/2}^{t_o + T_1/2} e^{in\,\Delta\omega\,\tau}\,d\tau = 0 \tag{17.5}$$

or alternately in discrete form as that

$$\sum_{j=0}^{N-1} e^{i2\pi nj/N} = 0. \tag{17.6}$$

Because the product $\Delta\omega\, T_1 = 2\pi$ relates $\Delta\omega$ to T_1, the symbols $\Delta\omega$ and T_1 together represent in (17.3) and (17.5) not two but only one parameter; you cannot set them independently, for the one merely inverts the other. If T_1 bears physical units then these will typically be units of time (seconds, for instance), whereupon $\Delta\omega$ will bear the corresponding units of angular frequency (such as radians per second). The frame offset t_o and the dummy variable τ must have the same dimensions[4] T_1 has. This matter is discussed further in § 17.2.

To prove (17.5) symbolically is easy: one merely carries out the indicated integration. To prove (17.6) symbolically is not much harder: one replaces the complex exponential $e^{i2\pi nj/N}$ by $\lim_{\epsilon\to 0^+} e^{(i-\epsilon)2\pi nj/N}$ and then uses (2.36) to evaluate the summation. Notwithstanding, we can do better, for an alternate, more edifying, physically more insightful explanation of the two equations is possible as follows. Because n is a nonzero integer, (17.5) and (17.6) represent sums of steps in every direction—that is, steps in every phase—in the Argand plane (more precisely, eqn. 17.6 represents a sum over

[2]That $2 \leq N$ is a redundant requirement, since (17.4)'s other lines imply it, but it doesn't hurt to state it anyway.

[3]An expression like $t_o \pm T_1/2$ means $t_o \pm (T_1/2)$, here and elsewhere in the book.

[4]The term *dimension* in this context refers to the kind of physical unit. A quantity like T_1 for example, measurable in seconds or years (but not, say, in kilograms or dollars), has dimensions of time. An automobile's speed having dimensions of length divided by time can be expressed in miles per hour as well as in meters per second but not directly, say, in volts per centimeter; and so on.

a discrete but balanced, uniformly spaced selection of phases). An appeal to symmetry forbids such sums from favoring any one phase $n\,\Delta\omega\,\tau$ or $2\pi nj/N$ over any other. This being the case, how could the sums of (17.5) and (17.6) come to any totals other than zero? The plain answer is that they can come to no other totals. A step in every direction is indeed no step at all. This is why (17.5) and (17.6) are so.[5]

We have actually already met Parseval's principle, informally, in § 9.7.2. There is also *Parseval's theorem* to come in § 18.2.8.

One can translate Parseval's principle from the Argand realm to the analogous realm of geometrical vectors, if needed, in the obvious way.

17.2 Time, space and frequency

A *frequency* is the inverse of an associated period of time, expressing the useful concept of the rate at which a cycle repeats. For example, an internal-combustion engine whose crankshaft revolves once every 20 milliseconds—which is to say, once every 1/3000 of a minute—runs thereby at a frequency of 3000 revolutions per minute (RPM) or, in other words, 3000 cycles per minute. Frequency however comes in two styles: cyclic frequency (as in the engine's example), conventionally represented by letters like ν and f; and angular frequency, by letters like ω and k. If T, ν and ω are letters taken to stand respectively for a period of time, for the associated cyclic frequency, and for the associated angular frequency, then by definition

$$\begin{aligned} \nu T &= 1, \\ \omega T &= 2\pi, \\ \omega &= 2\pi\nu. \end{aligned} \tag{17.7}$$

The period T will bear units of time like seconds or minutes. The cyclic frequency ν will bear units of inverse time like cycles per second (hertz) or

[5]The writer unfortunately knows of no conventionally established name for Parseval's principle. The name *Parseval's principle* seems as apt as any and this is the name the book will use.

A pedagogical knot seems to tangle Marc-Antoine Parseval's various namesakes. Because Parseval's principle can be extracted as a special case from Parseval's theorem (eqn. 18.45 in the next chapter), the literature sometimes indiscriminately applies the name "Parseval's theorem" to both. This is fine as far as it goes, but the knot arrives when one needs Parseval's principle to derive the Fourier series, which one needs to derive the Fourier transform, which one needs in turn to derive Parseval's theorem, at least as this book develops them. The way to untie the knot is to give Parseval's principle its own name and to let it stand as an independent result.

cycles per minute.[6] The angular frequency ω will bear units of inverse time like radians per second or radians per minute.

The last brings us to a point that has confused many students of science and engineering: if $T = 20$ ms and $\nu = 3000$ cycles/min, then why not $\nu T = (3000 \text{ cycles/min})(20 \text{ ms}) = 6.0 \times 10^4 \text{ cycle} \cdot \text{ms/min} = 1 \text{ cycle} \neq 1$? The answer is that a cycle is not conventionally held to be a unit of measure and, thus, does not conventionally enter into the arithmetic.[7] Minimally correct usage is rather that

$$\nu T = (3000 \,\text{min}^{-1})(20 \,\text{ms}) = 6.0 \times 10^4 \frac{\text{ms}}{\text{min}} = 1.$$

As for the word "cycle," one can include the word in the above line if one wishes to include it, but with the understanding that the word lacks arithmetical significance. Arithmetically, one can drop the word at any stage.

It follows, perhaps unexpectedly, that the cycle per minute and the radian per minute do not arithmetically differ from one another. Arithmetically, counterintuitively,

$$\frac{1}{60}\,\frac{\text{cycle}}{\text{second}} = 1\frac{\text{cycle}}{\text{minute}} = 1\frac{\text{radian}}{\text{minute}} = \frac{1}{60}\,\frac{\text{radian}}{\text{second}}.$$

This looks obviously wrong, of course, but don't worry: it is a mere tautology which, though perhaps literally accurate (we will explain why), expresses no very profound idea. Its sole point is that both the cycle per minute and the radian per minute, interpreted as units of measure, are units of $[\text{minute}]^{-1}$; whereas—in the context of phrases like "cycle per minute" and "radian per minute"—the words "cycle" and "radian" are verbal cues that, in and of themselves, play no actual part in the mathematics. *This is not because the cycle and the radian were ephemeral but rather because the minute is unfundamental.*

The minute, a unit of measure representing a definite but arbitrary quantity of time, requires arithmetical representation. The cycle and the radian,

[6]Notice incidentally, contrary to the improper verbal usage one sometimes hears, that there is no such thing as a "hert." Rather, "Hertz" is somebody's name. The uncapitalized form "hertz" thus is singular as well as plural.

[7]The writer believes the convention to be wise. The reason behind the convention is not easy to articulate (though the narrative will try to articulate it, anyway), but experience does seem to support the convention nevertheless. Little is gained, and convenience is lost, when one—contrary to convention—treats a countable entity like a cycle as one would treat an arbitrary quantity of physical reference like a second. The cycle and the second are not things of the same kind. As such, they tend not to approve treatment of the same kind, even if such treatment is possible.

by contrast, are nonarbitrary, discrete, inherently countable things; and, where things are counted, it is ultimately up to the mathematician to interpret the count (consider for instance that nine baseball caps may imply nine baseball players and one baseball team, but that there is nothing in the number nine itself to tell us so). To distinguish angular frequencies from cyclic frequencies, it remains to the mathematician to lend factors of 2π where needed.

If, nevertheless, you think the last displayed equation just too weird, then don't write it that way; but think of, say, gear ratios. A gear ratio might be 3:1 or 5:1 or whatever, but the ratio is unitless. You can *say* "3.0 turns of the small gear per turn of the large gear," but this manner of speaking does not make the "turn of the small gear per turn of the large gear" a proper unit of measure. The "cycle per radian" is in the same sense likewise not a proper unit of measure. (Now, if you *still* think the last displayed equation just too weird—well, it *is* weird. You can ignore the equation, instead interpreting the expression "cycle per radian" as a way of naming the number 2π. This sort of works, but beware that such an interpretation does not extend very well to other nonunit units like "decibel" and is not the interpretation the writer recommends. Also beware: an expression like $\sin\{[2\pi/4] \text{ radians}\} = \sin[2\pi/4] = 1$ means something sensible whereas an expression like $\sin\{[2\pi/4] \text{ dollars}\} = ??$ probably does not. Anyway, if "radian" is taken to be 1—as it must be taken if $\sin\{[2\pi/4] \text{ radians}\}$ is to come out right—then "cycle" must be taken to be 2π, which does not quite square with eqn. 17.7, does it? No, the problem is that the radian and the cycle are no units of measure.)[8]

The word "frequency" without a qualifying adjective is usually taken in English to mean cyclic frequency unless the surrounding context implies

[8]Some recent undergraduate engineering textbooks have taken to the style of

$$E = \frac{Q}{Cd} \text{ [volts/meter]}.$$

The intent seems to be to encourage undergraduates to include units of measure with their engineering quantities, as

$$E = \frac{Q}{Cd} = 5.29 \text{ volts/meter}.$$

Unfortunately, my own, occasional experience at teaching undergraduates suggests that undergraduates tend to read the textbook as though it had read

$$E = \left[\frac{Q}{Cd}\right] \left[1.0 \, \frac{\text{volt}}{\text{meter}}\right],$$

which is wrong and whose resultant confusion compounds, wasting hours of the undergraduates' time. It seems to me preferable to insist that undergraduates learn from the otherwise. Notwithstanding, interestingly, experience seems to find angular frequency to be oftener the more natural or convenient to use (but see § 19.7).

Frequencies exist in space as well as in time:

$$k\lambda = 2\pi. \tag{17.8}$$

Here, λ is a *wavelength* measured in meters or other units of length. The *wave number*[9] k is an angular spatial frequency measured in units like radians per meter; that is, $[\text{meter}]^{-1}$. (Oddly, no conventional symbol for *cyclic* spatial frequency seems to be current. The literature mostly just uses $k/2\pi$ which, in light of the potential for confusion between ν and ω in the temporal domain, is probably for the best.)

Where a wave propagates the propagation speed

$$v = \frac{\lambda}{T} = \frac{\omega}{k} \tag{17.9}$$

relates periods and frequencies in space and time.

Now, we must admit that we fibbed when we said (or implied) that T had to have dimensions of time. Physically, that is the usual interpretation, but mathematically T (and T_1, t, t_o, τ, etc.) can bear any units and indeed is not required to bear units at all, a fact to which § 17.1 has alluded. The only mathematical requirement is that the product $\omega T = 2\pi$ (or $\Delta\omega\, T_1 = 2\pi$ or the like, as appropriate) be dimensionless. However, when T has dimensions of length rather than of time it is conventional—indeed, it is practically mandatory if one wishes to be understood—to change $\lambda \leftarrow T$ and $k \leftarrow \omega$ as

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first the correct meaning of an unadorned equation like

$$E = \frac{Q}{Cd},$$

where, say, $Q = 13.3$ volt · sec/ohm, $C = 0.470$ sec/ohm, and $d = 5.35$ cm; and that they grasp the need not to write algebraically perplexing falsities such as that "d cm = 5.35 cm"—perplexing falsities which, unfortunately, the textbook style in question inadvertently encourages them to write.

When during an engineering lecture it becomes pedagogically necessary to associate units of measure to a symbolic equation, my own practice at the blackboard has been to write

$$E = \frac{Q}{Cd}, \quad E : [\text{volts/meter}].$$

Done sparingly, this seems to achieve the desired effect, though in other instances the unadorned style is preferred. —THB—

[9] One could wish for a better name for the thing than *wave number.* By whatever name, the wave number k is no integer, notwithstanding that the letter k tends to represent integers in other contexts.

this section has done, though the essential Fourier mathematics is the same regardless of T's dimensions (if any) or of whether alternate symbols like λ and k are used.

17.3 Some symmetrical pulses of unit area

The Dirac delta of § 7.7 and Fig. 7.11 is useful among other reasons for the unit area it covers, but for some purposes its curve is too sharp. This section introduces several alternate pulses each of unit area. Each pulse is symmetrical. Each is less sharp. Applications can substitute any of them for the Dirac delta—or use any in the limit to implement the Dirac delta—as need arises.

17.3.1 The basic nonanalytic pulses

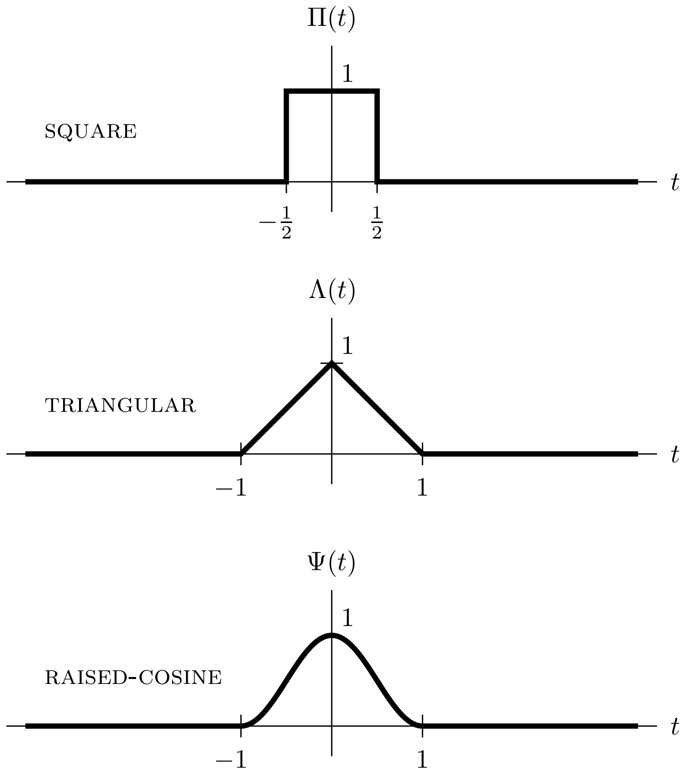
The *square, triangular* or *raised-cosine pulse* of Fig. 17.3,

$$\begin{aligned}
\Pi(t) &\equiv \begin{cases} 1 & \text{if } |t| < 1/2, \\ 1/2 & \text{if } |t| = 1/2, \\ 0 & \text{otherwise;} \end{cases} \\
\Lambda(t) &\equiv \begin{cases} 1 - |t| & \text{if } |t| \le 1, \\ 0 & \text{otherwise;} \end{cases} \\
\Psi(t) &\equiv \begin{cases} [1 + \cos(\pi t)]/2 & \text{if } |t| \le 1, \\ 0 & \text{otherwise;} \end{cases} \\
\Im(t) &= 0;
\end{aligned} \tag{17.10}$$

substitutes for or implements the Dirac delta. Each pulse evidently shares Dirac's property that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{T}\,\delta\left(\frac{\tau - t_o}{T}\right) d\tau &= 1, \\
\int_{-\infty}^{\infty} \frac{1}{T}\Pi\left(\frac{\tau - t_o}{T}\right) d\tau &= 1, \\
\int_{-\infty}^{\infty} \frac{1}{T}\Lambda\left(\frac{\tau - t_o}{T}\right) d\tau &= 1, \\
\int_{-\infty}^{\infty} \frac{1}{T}\Psi\left(\frac{\tau - t_o}{T}\right) d\tau &= 1,
\end{aligned} \tag{17.11}$$

Figure 17.3: The basic nonanalytic pulses.



for any real $T > 0$ and real t_o. In the limit,

$$\begin{aligned}\lim_{T\to 0^+} \frac{1}{T}\Pi\left(\frac{t-t_o}{T}\right) &= \delta(t-t_o),\\ \lim_{T\to 0^+} \frac{1}{T}\Lambda\left(\frac{t-t_o}{T}\right) &= \delta(t-t_o),\\ \lim_{T\to 0^+} \frac{1}{T}\Psi\left(\frac{t-t_o}{T}\right) &= \delta(t-t_o).\end{aligned} \tag{17.12}$$

The three basic nonanalytic pulses can do more than to implement Dirac. The three share the convenient property for all t that

$$\sum_{m=-\infty}^{\infty} \Pi(t-m) = \sum_{m=-\infty}^{\infty} \Lambda(t-m) = \sum_{m=-\infty}^{\infty} \Psi(t-m) = 1, \tag{17.13}$$

a property that makes the three especially useful in the rendering of discretely sampled electronic signals and the like (see also § 19.2). Related is the property that

$$\Pi\left(\pm\frac{1}{2}\right) = \Lambda\left(\pm\frac{1}{2}\right) = \Psi\left(\pm\frac{1}{2}\right) = \frac{1}{2}, \tag{17.14}$$

though for the square $\Pi(\pm 1/2)$, admittedly, the specific value one imputes to the discontinuity is a matter of interpretation. Significant too is the property that

$$\Pi(t) = \Lambda(t) = \Psi(t) = 0 \;\text{ for all } |t| \geq 1. \tag{17.15}$$

Indeed,

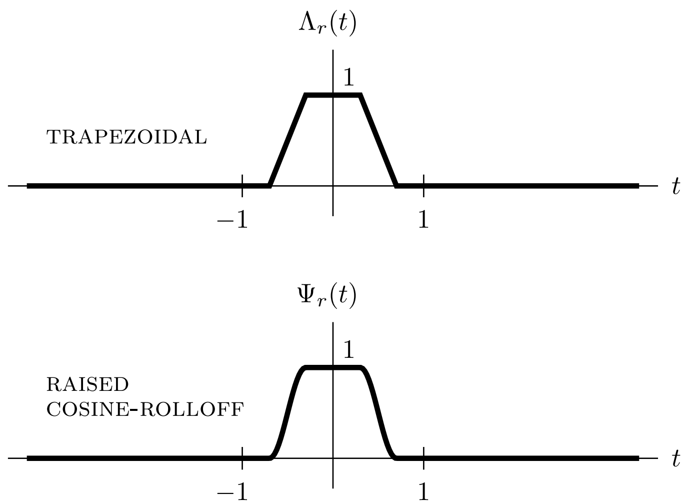
$$\Pi(t) = 0 \;\text{ for all } |t| \geq \frac{1}{2}. \tag{17.16}$$

17.3.2 Rolloff pulses

No other pulse that satisfies (17.13) and the other properties of § 17.3.1 is so compact[10] as the simple square pulse $\Pi(t)$ of (17.10) and Fig. 17.3. The other basic nonanalytic pulses, $\Lambda(t)$ and $\Psi(t)$, of (17.10) and Fig. 17.3 act over the domain $|t| < 1$ as (17.15) has observed; whereas $\Pi(t)$ per (17.16) confines itself to a minimal $|t| < 1/2$. The abruptness of $\Pi(t)$ can cause trouble, though. Some applications would prefer a smoother compromise.

[10] Pure mathematics might add a proof of this fact but we shall take it as obvious.

Figure 17.4: Rolloff pulses.



The *rolloff pulses*

$$
\begin{aligned}
\Lambda_r(t) &\equiv \begin{cases} 1 & \text{if } |t| \le (1-r)/2, \\ \dfrac{1}{2} + \dfrac{\frac{1}{2} - |t|}{r} & \text{if } \dfrac{1-r}{2} \le |t| \le \dfrac{1+r}{2}, \\ 0 & \text{otherwise;} \end{cases} \\
\Psi_r(t) &\equiv \begin{cases} 1 & \text{if } |t| \le (1-r)/2, \\ \dfrac{1}{2} + \dfrac{1}{2}\sin\left(\pi \dfrac{\frac{1}{2} - |t|}{r}\right) & \text{if } \dfrac{1-r}{2} \le |t| \le \dfrac{1+r}{2}, \\ 0 & \text{otherwise;} \end{cases} \\
&\Im(t) = 0,\ \Im(r) = 0,\ 0 \le r \le 1;
\end{aligned}
\tag{17.17}
$$

of Fig. 17.4 afford the compromise. Narrower than the full $\Lambda(t)$ and $\Psi(t)$, $\Lambda_r(t)$ and $\Psi_r(t)$ nevertheless avoid the abrupt jumps of the square $\Pi(t)$. The $\Psi_r(t)$ avoids corners , too. The rolloff parameter r quantifies the compromise: the nearer r is to 0, the more like the square pulse; the nearer r

is to 1, the more like the triangular or raised-cosine pulse.[11] Indeed,

$$\begin{aligned}\Psi_0(t) = \Lambda_0(t) &= \Pi(t),\\ \Lambda_1(t) &= \Lambda(t),\\ \Psi_1(t) &= \Psi(t).\end{aligned} \tag{17.18}$$

If $0 < r < 1$, then a rolloff pulse resembles the square pulse by maintaining a height of 1 in its central region but softens the square pulse's abrupt transition by rolling off, smoothly, along a triangular or raised-cosine track at the edge. Though nonanalytic and time-limited, the trapezoidal and raised cosine-rolloff pulses are continuous and the raised cosine-rolloff pulse even has a continuous first derivative, properties that make the pulse useful in applications that would have preferred a true square pulse but cannot quite tolerate the abruptness of the square pulse's transition.

Figure 17.5 takes a closer look at the raised cosine-rolloff pulse.

Properties (17.11) through (17.14) all apply to the two rolloff pulses. As for (17.15),

$$\Lambda_r(t) = \Psi_r(t) = 0 \ \text{ for all } |t| \geq (1+r)/2, \tag{17.19}$$

which is better.

17.3.3 The Gaussian pulse (preview)

Looking ahead, if we may further abuse the Greek capitals to let them represent pulses whose shapes they accidentally resemble, then a subtler implementation of Dirac's delta—more complicated to handle but analytic (§ 8.4) and therefore preferable for some purposes—is the *Gaussian pulse,*

$$\begin{aligned}\lim_{T\to 0^+} \frac{1}{T}\Omega\left(\frac{t-t_o}{T}\right) &= \delta(t-t_o), \\ \Omega(t) &\equiv \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right),\end{aligned} \tag{17.20}$$

of Fig. 17.6, the mathematics of which § 18.4 and chapter 20 will begin to unfold.

Equation (18.59) will later find that the Gaussian pulse $\Omega(t)$ covers unit area as this section's other pulses do. Most of the section's other properties do not however apply to the Gaussian. For example, the Gaussian does not wholly vanish at large t (though it almost does). Interestingly, unlike the several nonanalytic pulses, the Gaussian pulse is sensibly defined for complex t.

[11] [37]

Figure 17.5: A closer look at the raised cosine-rolloff pulse.

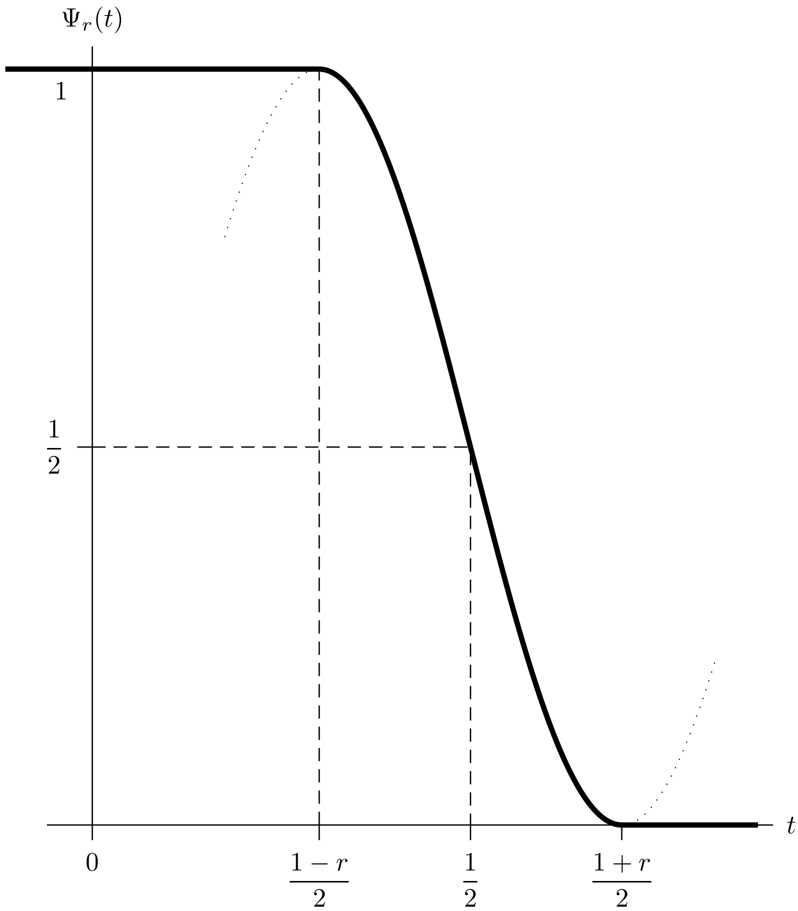
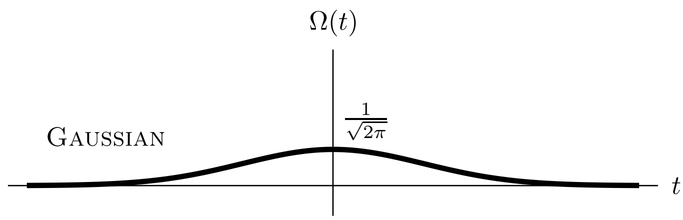


Figure 17.6: The Gaussian pulse.



17.4 Expanding repeating waveforms in Fourier series

The Fourier series represents a repeating waveform (17.1) as a superposition of sinusoids. More precisely, inasmuch as Euler's formula (5.18) renders each sinusoid as the sum of two *complex exponentials,* the Fourier series represents a repeating waveform as a superposition

$$f(t) = \sum_{j=-\infty}^{\infty} a_j e^{ij\,\Delta\omega\, t} \tag{17.21}$$

of complex exponentials in which (17.3) is obeyed and yet neither the several Fourier coefficients a_j nor the waveform $f(t)$ itself need be real. Whether one can properly represent every repeating waveform as a superposition (17.21) of complex exponentials is a question §§ 17.4.4 and 17.7 will address later; but, at least to the extent to which one can properly represent such a waveform, we will now assert that one can recover any or all of the waveform's Fourier coefficients a_j by choosing an arbitrary frame offset t_o ($t_o = 0$ being a typical choice) and then integrating

$$a_j = \frac{1}{T_1} \int_{t_o - T_1/2}^{t_o + T_1/2} e^{-ij\,\Delta\omega\,\tau} f(\tau)\, d\tau. \tag{17.22}$$

17.4.1 Derivation of the Fourier-coefficient formula

But why should (17.22) work? How is it to recover a Fourier coefficient a_j? The answer is that it recovers a Fourier coefficient a_j by isolating it, and that it isolates it by shifting frequencies and integrating.

Equation (17.21) has proposed to express a repeating waveform as a series of complex exponentials, each exponential of the form $a_j e^{ij\,\Delta\omega\, t}$ in which a_j is a weight to be determined. Unfortunately, (17.21) can hardly be very useful until the several a_j are determined, whereas how to determine a_j from (17.21) for a given value of j is not immediately obvious.

The trouble with using (17.21) to determine the several coefficients a_j is that it includes all the terms of the series and, hence, all the coefficients a_j at once. To determine a_j for a given value of j, one should like to suppress the entire series except the single element $a_j e^{ij\,\Delta\omega\, t}$, isolating this one element for analysis. Fortunately, Parseval's principle (17.5) gives us a way to do this, as we shall soon see.

Now, to prove (17.22) we mean to use (17.22), a seemingly questionable act. Nothing prevents us however from taking only the right side of (17.22)—not as an equation but as a mere expression—and doing some algebra with it to see where the algebra leads, for if the algebra should lead to the *left* side of (17.22) then we should have proven the equation. Accordingly, changing dummy variables $\tau \leftarrow t$ and $\ell \leftarrow j$ in (17.21) and then substituting into (17.22)'s right side the resulting expression for $f(\tau)$, we have by successive steps that

$$
\begin{aligned}
&\frac{1}{T_1}\int_{t_o-T_1/2}^{t_o+T_1/2} e^{-ij\,\Delta\omega\,\tau} f(\tau)\,d\tau \\
&\quad= \frac{1}{T_1}\int_{t_o-T_1/2}^{t_o+T_1/2} e^{-ij\,\Delta\omega\,\tau} \sum_{\ell=-\infty}^{\infty} a_\ell e^{i\ell\,\Delta\omega\,\tau}\,d\tau \\
&\quad= \frac{1}{T_1}\sum_{\ell=-\infty}^{\infty} a_\ell \int_{t_o-T_1/2}^{t_o+T_1/2} e^{i(\ell-j)\,\Delta\omega\,\tau}\,d\tau \\
&\quad= \frac{a_j}{T_1}\int_{t_o-T_1/2}^{t_o+T_1/2} e^{i(j-j)\,\Delta\omega\,\tau}\,d\tau \\
&\quad= \frac{a_j}{T_1}\int_{t_o-T_1/2}^{t_o+T_1/2} d\tau = a_j,
\end{aligned}
$$

in which Parseval's principle (17.5) has killed all but the $\ell = j$ term in the summation. Thus is (17.22) proved.

Except maybe to the extent to which one would like to examine convergence (see the next paragraph), the idea behind the proof remains more interesting than the proof itself, for one would like to know not only the fact that (17.22) is true but also the thought which leads one to propose the equation in the first place. The thought is as follows. Assuming that (17.21) can indeed represent the waveform $f(t)$ properly, one observes that the transforming factor $e^{-ij\,\Delta\omega\,\tau}$ of (17.22) serves to shift the waveform's jth component $a_j e^{ij\,\Delta\omega\,t}$—whose angular frequency is evidently $\omega = j\,\Delta\omega$—down to a frequency of zero, incidentally shifting the waveform's several other components to various nonzero frequencies as well. Significantly, the transforming factor leaves each shifted frequency to be a whole multiple of the waveform's fundamental frequency $\Delta\omega$. By Parseval's principle, (17.22)'s integral then kills all the thus frequency-shifted components except the zero-shifted one *by integrating the components over complete cycles,* passing only the zero-shifted component which, once shifted, has no cycle. Such is the thought which has given rise to the equation.

Before approving the proof's interchange of summation and integration, a pure mathematician would probably have preferred to establish conditions under which the summation and integration should each converge. To the applied mathematician however, the establishment of *general* conditions turns out to be an unrewarding exercise,[12] so we will let the matter pass with this remark: nothing prevents one from treating (17.21) as

$$f(t) = \lim_{J\to\infty} \sum_{j=-J}^{J} a_j e^{ij\,\Delta\omega\,t},$$

which manages the convergence problem (to the extent to which it even is a problem) in most cases of practical interest. Further work on the convergence problem is left to the charge of the concerned reader, but see also §§ 7.3.4, 7.3.5 and 22.4.

[12]The conditions conventionally observed among professional mathematicians seem to be known as the *Dirichlet conditions.* As far as this writer can tell, the Dirichlet conditions lie pretty distant from applications—not that there aren't concrete applications that transgress them (for example in stochastics), but rather that the failure of (17.22) to converge in a given concrete application is more readily apparent by less abstract means than Dirichlet's.

This book could merely *list* the Dirichlet conditions without proof; but, since the book is a book of derivations, it will decline to do that. The conditions look plausible. We'll leave it at that.

The writer suspects that few readers will ever encounter a concrete application that really wants the Dirichlet conditions, but one never knows. The interested reader can pursue Dirichlet elsewhere. (Where? No recommendation. No book on the writer's shelf seems strong enough on Dirichlet to recommend.)

17.4.2 The square wave

According to (17.22), the Fourier coefficients of Fig. 17.1's square wave are, if $t_o = T_1/4$ is chosen and by successive steps,

$$\begin{aligned} a_j &= \frac{1}{T_1}\int_{-T_1/4}^{3T_1/4} e^{-ij\,\Delta\omega\,\tau} f(\tau)\,d\tau \\ &= \frac{A}{T_1}\left[\int_{-T_1/4}^{T_1/4} - \int_{T_1/4}^{3T_1/4}\right] e^{-ij\,\Delta\omega\,\tau}\,d\tau \\ &= \frac{iA}{2\pi j} e^{-ij\,\Delta\omega\,\tau}\left[\Big|_{-T_1/4}^{T_1/4} - \Big|_{T_1/4}^{3T_1/4}\right]. \end{aligned}$$

But

$$\begin{aligned} e^{-ij\,\Delta\omega\,\tau}\Big|_{\tau=-T_1/4} = e^{-ij\,\Delta\omega\,\tau}\Big|_{\tau=3T_1/4} &= i^j, \\ e^{-ij\,\Delta\omega\,\tau}\Big|_{\tau=T_1/4} &= (-i)^j, \end{aligned}$$

so

$$\begin{aligned} &e^{-ij\,\Delta\omega\,\tau}\left[\Big|_{-T_1/4}^{T_1/4} - \Big|_{T_1/4}^{3T_1/4}\right] \\ &\quad= [(-i)^j - i^j] - [i^j - (-i)^j] = 2[(-i)^j - i^j] \\ &\quad= \ldots, -i4, 0, i4, 0, -i4, 0, i4, \ldots \text{ for } j = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \end{aligned}$$

Therefore,

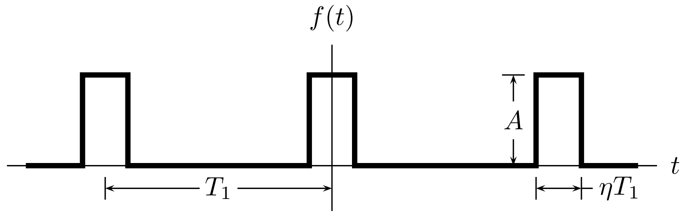
$$\begin{aligned} a_j &= \left[(-i)^j - i^j\right]\frac{i2A}{2\pi j} \\ &= \begin{cases} (-)^{(j-1)/2} 4A/2\pi j & \text{for odd } j, \\ 0 & \text{for even } j, \end{cases} \end{aligned} \tag{17.23}$$

are the square wave's Fourier coefficients which, when the coefficients are applied to (17.21) and when (5.18) is invoked, indeed yield the specific series of sinusoids (17.2) and Fig. 17.2 have proposed.

17.4.3 The rectangular pulse train

The square wave of § 17.4.2 is an important, canonical case and (17.2) is arguably worth memorizing. After the square wave, however, an endless

Figure 17.7: A rectangular pulse train.



variety of repeating waveforms present themselves. Section 17.4.2 has exampled how to compute their Fourier series.

One variant on the square wave is nonetheless interesting enough to attract special attention. This variant is the *pulse train* of Fig. 17.7,

$$f(t) = A \sum_{j=-\infty}^{\infty} \Pi\left(\frac{t - jT_1}{\eta T_1}\right); \tag{17.24}$$

where $\Pi(\cdot)$ is the square pulse of (17.10); the symbol A represents the pulse's full height rather than the half height of Fig. 17.1; and the dimensionless factor $0 \le \eta \le 1$ is the train's *duty cycle,* the fraction of each cycle its pulse is as it were on duty. By the routine of § 17.4.2,

$$\begin{aligned} a_j &= \frac{1}{T_1}\int_{-T_1/2}^{T_1/2} e^{-ij\,\Delta\omega\,\tau} f(\tau)\,d\tau \\ &= \frac{A}{T_1}\int_{-\eta T_1/2}^{\eta T_1/2} e^{-ij\,\Delta\omega\,\tau}\,d\tau \\ &= \frac{iA}{2\pi j} e^{-ij\,\Delta\omega\,\tau}\Big|_{-\eta T_1/2}^{\eta T_1/2} = \frac{2A}{2\pi j}\sin\frac{2\pi\eta j}{2} \end{aligned}$$

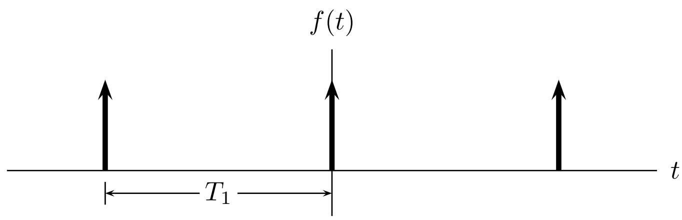
for $j \neq 0$. On the other hand,

$$a_0 = \frac{1}{T_1}\int_{-T_1/2}^{T_1/2} f(\tau)\,d\tau = \frac{A}{T_1}\int_{-\eta T_1/2}^{\eta T_1/2} d\tau = \eta A$$

is the waveform's mean value. Altogether for the pulse train,

$$a_j = \begin{cases} \dfrac{2A}{2\pi j}\sin\dfrac{2\pi\eta j}{2} & \text{if } j \neq 0, \\ \eta A & \text{if } j = 0 \end{cases} \tag{17.25}$$

Figure 17.8: A Dirac delta pulse train.



(though eqn. 17.41 will improve the notation later).

An especially interesting special case occurs when the duty cycle grows very short. Since $\lim_{\eta\to 0^+} \sin(2\pi\eta j/2) = 2\pi\eta j/2$ according to (8.32), it follows from (17.25) that

$$\lim_{\eta\to 0^+} a_j = \eta A, \tag{17.26}$$

the same for every index j. As the duty cycle η tends to vanish the pulse tends to disappear and the Fourier coefficients along with it; but we can compensate for vanishing duty if we wish by increasing the pulse's amplitude A proportionally, maintaining the product

$$\eta T_1 A = 1 \tag{17.27}$$

of the pulse's width ηT_1 and its height A—and thus preserving unit area[13] under the pulse. In the limit $\eta \to 0^+$, the pulse then by definition becomes the Dirac delta of Fig. 7.11, and the pulse train by construction becomes the *Dirac delta pulse train* of Fig. 17.8. Enforcing (17.27) on (17.26) yields the Dirac delta pulse train's Fourier coefficients

$$a_j = \frac{1}{T_1}. \tag{17.28}$$

[13]In light of the discussion of time, space and frequency in § 17.2, we should clarify that we do not here mean a physical area measurable in square meters or the like. We merely mean the dimensionless product of the width (probably measured in units of time like seconds) and the height (correspondingly probably measured in units of frequency like inverse seconds) of the rectangle a single pulse encloses in Fig. 17.7. Though it is not a physical area the rectangle one sketches on paper to represent it, as in the figure, of course does have an area. The word *area* here is meant in the latter sense.

Looking ahead, Tables 18.4 and 18.5 will tabulate the Fourier *transforms* of various functions. Equation (19.4) and the method of § 19.1.2 can convert the tabulated transforms to corresponding Fourier series at need.

17.4.4 Linearity and sufficiency

The Fourier series is evidently linear according to the rules of § 7.3.3. That is, if the Fourier coefficients of $f_1(t)$ are a_{j1} and the Fourier coefficients of $f_2(t)$ are a_{j2}, and if the two waveforms $f_1(t)$ and $f_2(t)$ share the same fundamental period T_1, then the Fourier coefficients of $f(t) = f_1(t) + f_2(t)$ are $a_j = a_{j1} + a_{j2}$. Likewise, the Fourier coefficients of $\alpha f(t)$ are αa_j and the Fourier coefficients of the null waveform $f_{\mathrm{null}}(t) \equiv 0$ are themselves null, thus satisfying the conditions of linearity.

All this however supposes that the Fourier series actually works.[14] Though Fig. 17.2 is suggestive, the figure alone hardly serves to demonstrate that every repeating waveform were representable as a Fourier series. To try to consider every repeating waveform at once would be too much to try at first in any case, so let us start from a more limited question: does there exist any continuous, repeating waveform $f(t) \neq 0$ of period T_1 whose Fourier coefficients $a_j = 0$ are identically zero? [By $f(t) \neq 0$, we here mean[15] that at least one real value of t is to exist for which $f(t) \neq 0$. In logical notation,[16] $\exists t \in \mathbb{R} : f(t) \neq 0$.]

If the waveform $f(t)$ in question is continuous then nothing prevents us from discretizing (17.22) as

$$\begin{aligned} a_j &= \lim_{M\to\infty} \frac{1}{T_1} \sum_{\ell=-M}^{M} e^{(-ij\,\Delta\omega)(t_o+\ell\,\Delta\tau_M)} f(t_o + \ell\,\Delta\tau_M)\,\Delta\tau_M, \\ \Delta\tau_M &\equiv \frac{T_1}{2M+1}, \end{aligned}$$

and further discretizing the waveform itself as

$$f(t) = \lim_{M\to\infty} \sum_{p=-\infty}^{\infty} f(t_o + p\,\Delta\tau_M)\,\Pi\left[\frac{t-(t_o+p\,\Delta\tau_M)}{\Delta\tau_M}\right],$$

[14] The remainder of this dense subsection can be regarded as optional reading.

[15] We generally mean the same elsewhere in the book, too.

[16] It is unnecessary to read logical notation to read this book but you might wish to learn a little of it, anyway. Besides $\exists x :$, "at least one x exists such that ...," you also have $\exists! x :$, "exactly one x exists such that ...," $\nexists x :$, "no x exists such that ...," and $\forall x :$, "for all x ...." Along with such symbols are also $\wedge$, $\vee$ and $\neg$, which respectively mean "and," "or" and "not." The symbol $\mathbb{R}$ represents the real domain.

in which $\Pi[\cdot]$ is the square pulse of (17.10). Substituting the discretized waveform into the discretized formula for a_j, we have that

$$\begin{aligned} a_j &= \lim_{M\to\infty} \frac{\Delta\tau_M}{T_1} \sum_{\ell=-M}^{M} \sum_{p=-\infty}^{\infty} e^{(-ij\,\Delta\omega)(t_o+\ell\,\Delta\tau_M)} f(t_o + p\,\Delta\tau_M)\Pi(\ell - p) \\ &= \lim_{M\to\infty} \frac{\Delta\tau_M}{T_1} \sum_{\ell=-M}^{M} e^{(-ij\,\Delta\omega)(t_o+\ell\,\Delta\tau_M)} f(t_o + \ell\,\Delta\tau_M). \end{aligned}$$

If we define the $(2M+1)$-element vectors and $(2M+1)\times(2M+1)$ matrix

$$\begin{aligned} [\mathbf{f}_M]_\ell &\equiv f(t_o + \ell\,\Delta\tau_M), \\ [\mathbf{a}_M]_j &\equiv a_j, \\ [C_M]_{j\ell} &\equiv \frac{\Delta\tau_M}{T_1} e^{(-ij\,\Delta\omega)(t_o+\ell\,\Delta\tau_M)}, \\ -M \le (j,\ell) &\le M, \end{aligned}$$

then matrix notation renders the last equation as

$$\lim_{M\to\infty} \mathbf{a}_M = \lim_{M\to\infty} C_M \mathbf{f}_M,$$

whereby

$$\lim_{M\to\infty} \mathbf{f}_M = \lim_{M\to\infty} C_M^{-1} \mathbf{a}_M,$$

assuming that C_M is invertible.

But is C_M invertible? This seems a hard question to answer until we realize that the rows of C_M consist of sampled complex exponentials which repeat over the interval T_1 and thus stand subject to Parseval's principle (17.6). Realizing this, we can do better than merely to state that C_M is invertible: we can write down its actual inverse,

$$[C_M^{-1}]_{\ell j} = \frac{T_1}{(2M+1)\,\Delta\tau_M} e^{(+ij\,\Delta\omega)(t_o+\ell\,\Delta\tau_M)},$$

such that[17] $C_M C_M^{-1} = I_{-M}^{M}$ and thus per (13.2) also that $C_M^{-1} C_M = I_{-M}^{M}$. So, the answer to our question is that, yes, C_M is invertible.

[17] Equation (11.30) has defined the notation I_{-M}^{M}, representing a $(2M+1)$-dimensional identity matrix whose string of ones extends along its main diagonal from $j = \ell = -M$ through $j = \ell = M$.

Because C_M is invertible, § 14.2 has it that neither $\mathbf{f}_M$ nor $\mathbf{a}_M$ can be null unless both are. In the limit $M \to \infty$, this implies[18] that no continuous, repeating waveform $f(t) \neq 0$ exists whose Fourier coefficients $a_j = 0$ are identically zero.

Now consider a continuous, repeating waveform[19] $F(t)$ and its Fourier series $f(t)$. Let $\Delta F(t) \equiv F(t) - f(t)$ be the part of $F(t)$ unrepresentable as a Fourier series, continuous because both $F(t)$ and $f(t)$ are continuous. Being continuous and unrepresentable as a Fourier series, $\Delta F(t)$ has null Fourier coefficients; but as the last paragraph has concluded this can only be so if $\Delta F(t) = 0$. Hence, $\Delta F(t) = 0$ indeed, which implies[20] that $f(t) = F(t)$. In other words, *every continuous, repeating waveform is representable as a Fourier series.*

And what of discontinuous waveforms? Well, the square wave of Figs. 17.1 and 17.2 this chapter has posed as its principal example is a repeating waveform but, of course, not a continuous one. A truly discontinuous waveform would admittedly invalidate the discretization above of $f(t)$, but see: nothing prevents us from approximating the square wave's discontinuity by an arbitrarily steep slope (as in § 17.3.2), whereupon this subsection's conclusion again applies.[21]

The better, subtler, more complete answer to the question though is that a discontinuity incurs Gibbs' phenomenon, which § 17.7 will derive.

---

[18] Pure mathematics might have preferred an alternate proof that never discretized the function. A professional mathematician might construct such an alternate proof on § 18.2.8 and its (18.46) or (18.47), or might prefer yet a different approach such as the Cantor-Riemann approach outlined in [164].

Nevertheless, for purpose of applications, your writer prefers this subsection's approach, for it avoids overgeneralization of the problem.

[19] Chapter 18 will use the capital letter F to represent a transform (such as a Fourier or Laplace transform) of a function f. However, that is not the use here.

[20] Chapter 8's footnote 6 has argued in a similar style, earlier in the book.

[21] Where this subsection's conclusion cannot be made to apply is where unreasonable waveforms like $A \sin[B/\sin \omega t]$ come into play. We will leave to the professional mathematician the classification of such unreasonable waveforms, the investigation of the waveforms' Fourier series and the provision of greater rigor generally.

One can object to the subsection's reliance on discretization, yet discretization is a useful technique, and to the extent to which pure mathematics has not yet recognized and formalized it, *maybe* that suggests—in the spirit of [136]—that some interested professional mathematician has more work to do, whenever he gets around to it. Or maybe it doesn't. Meanwhile, a lengthy, alternate, more abstractly rigorous proof that does not appeal to discretization is found in [43, chapter 3].

17.4.5 The trigonometric form

It is usually best, or at least neatest and cleanest, and moreover most evocative, to express Fourier series in terms of complex exponentials as (17.21) and (17.22) do. When the repeating waveform $f(t)$ is real, though, to express the series in terms of sines and cosines instead can be an attractive alternative. Euler's formula (5.12) makes (17.21) to be

$$f(t) = a_0 + \sum_{j=1}^{\infty} \left[(a_j + a_{-j}) \cos j\,\Delta\omega\, t + i(a_j - a_{-j}) \sin j\,\Delta\omega\, t \right].$$

Then, superimposing coefficients in (17.22),

$$\begin{aligned}
a_0 &= \frac{1}{T_1} \int_{t_o - T_1/2}^{t_o + T_1/2} f(\tau)\, d\tau, \\
b_j \equiv \ (a_j + a_{-j}) &= \frac{2}{T_1} \int_{t_o - T_1/2}^{t_o + T_1/2} \cos(j\,\Delta\omega\,\tau) f(\tau)\, d\tau, \\
c_j \equiv i(a_j - a_{-j}) &= \frac{2}{T_1} \int_{t_o - T_1/2}^{t_o + T_1/2} \sin(j\,\Delta\omega\,\tau) f(\tau)\, d\tau,
\end{aligned} \tag{17.29}$$

which give the Fourier series the trigonometric form

$$f(t) = a_0 + \sum_{j=1}^{\infty} \left(b_j \cos j\,\Delta\omega\, t + c_j \sin j\,\Delta\omega\, t \right). \tag{17.30}$$

The complex conjugate of (17.22) is

$$a_j^* = \frac{1}{T_1} \int_{t_o - T_1/2}^{t_o + T_1/2} e^{+ij\,\Delta\omega\,\tau} f^*(\tau)\, d\tau.$$

If the waveform happens to be real then $f^*(t) = f(t)$, which in light of the last equation and (17.22) implies that

$$a_{-j} = a_j^* \text{ if } \Im[f(t)] = 0. \tag{17.31}$$

Combining (17.29) and (17.31), we have that

$$\left. \begin{aligned} b_j &= \ \ 2\Re(a_j) \\ c_j &= -2\Im(a_j) \end{aligned} \right\} \text{ if } \Im[f(t)] = 0. \tag{17.32}$$

17.5 Parseval, Poisson and Euler

The Fourier transform of § 17.4 opens some interesting prospects to Parseval's principle of § 17.1.

17.5.1 Parseval's equality

The product of a Fourier series and its conjugate is

$$\begin{aligned} f^*(t)f(t) &= \left[\sum_{k=-\infty}^{\infty} a_k^* e^{-ik\,\Delta\omega\,t}\right]\left[\sum_{j=-\infty}^{\infty} a_j e^{ij\,\Delta\omega\,t}\right] \\ &= \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} a_k^* a_j e^{i(j-k)\,\Delta\omega\,t}. \end{aligned}$$

Integrating over a single period,

$$\begin{aligned} \int_{t_o-T_1/2}^{t_o+T_1/2} f^*(\tau)f(\tau)\,d\tau &= \int_{t_o-T_1/2}^{t_o+T_1/2} \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} a_k^* a_j e^{i(j-k)\,\Delta\omega\,t}\,d\tau \\ &= \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} a_k^* a_j \int_{t_o-T_1/2}^{t_o+T_1/2} e^{i(j-k)\,\Delta\omega\,t}\,d\tau. \end{aligned}$$

Parseval's principle of § 17.1 nulls the last integration except when $j = k$, so

$$\int_{t_o-T_1/2}^{t_o+T_1/2} f^*(\tau)f(\tau)\,d\tau = T_1 \sum_{j=-\infty}^{\infty} a_j^* a_j.$$

Dividing by T_1,

$$\frac{1}{T_1}\int_{t_o-T_1/2}^{t_o+T_1/2} f^*(\tau)f(\tau)\,d\tau = \sum_{j=-\infty}^{\infty} a_j^* a_j; \tag{17.33}$$

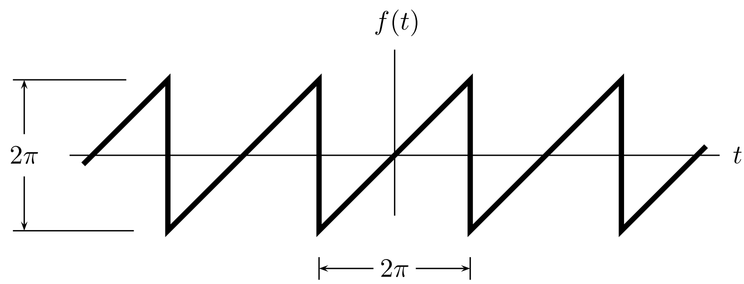
or, if you prefer,

$$\frac{1}{T_1}\int_{t_o-T_1/2}^{t_o+T_1/2} |f(\tau)|^2\,d\tau = \sum_{j=-\infty}^{\infty} |a_j|^2; \tag{17.34}$$

or even

$$\frac{1}{T_1}\int_{T_1} |f(\tau)|^2\,d\tau = \sum_{j=-\infty}^{\infty} |a_j|^2. \tag{17.35}$$

Figure 17.9: Poisson's ramp.



Whether as (17.33), (17.34) or (17.35), the result is *Parseval's equality*[22] which, as you see, connects a function's mean square to its Fourier coefficients.

Related to Parseval's equality are (18.44) and the results of § 18.2.8.

17.5.2 Poisson's ramp

Poisson's ramp,[23]

$$f(t) = \sum_{k=-\infty}^{\infty} (t - 2\pi k)\Pi\left(\frac{t - 2\pi k}{2\pi}\right), \tag{17.36}$$

is the repeating waveform of Fig. 17.9. Various applications including the Basel problem of § 17.5.3 use it. The method of § 17.4 finds the ramp's Fourier coefficients to be

$$a_j = \begin{cases} 0 & \text{if } j = 0; \\ -(-)^j/ij & \text{otherwise;} \end{cases} \tag{17.37}$$

[22][43, § 2.3]

[23]The source [188] names Poisson and develops Poisson's concept, thus earning credit for the idea here presented. However, that source does not pair the specific words "Poisson's ramp" as far as the author of the book you are reading is aware. The book will take responsibility for the words.

as one can show—observing that $\Delta\omega = 2\pi/T_1 = 1$—via Table 9.1 by

$$\begin{aligned} a_j &= \frac{1}{2\pi}\int_{-\pi}^{\pi} te^{-ijt}\,dt = \frac{e^{-ijt}}{2\pi}\left(\frac{1}{j^2} - \frac{t}{ij}\right)\bigg|_{t=-\pi}^{\pi} \\ &= \frac{e^{-ijt}}{2\pi}\left(\frac{1+ijt}{j^2}\right)\bigg|_{t=-\pi}^{\pi} = \frac{(-)^j}{2\pi}\left(\frac{ij2\pi}{j^2}\right) = -\frac{(-)^j}{ij}. \end{aligned}$$

17.5.3 Euler and the Basel problem

Substituting (17.37) into (17.34) with $t_o = 0$, using the period $T_1 = 2\pi$ of (17.36) and of Fig. 17.9,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\tau)|^2\,d\tau = \sum_{j\neq 0}\frac{1}{j^2} = 2\sum_{j=1}^{\infty}\frac{1}{j^2}.$$

On the other hand, by Fig. 17.9 and Table 7.1, via elementary integration without Fourier's help,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\tau)|^2\,d\tau = \frac{(2\pi)^2}{\mathrm{0xC}}.$$

Combining the last two equations and rearranging factors,

$$\sum_{j=1}^{\infty}\frac{1}{j^2} = \frac{(2\pi)^2}{\mathrm{0x18}}, \tag{17.38}$$

a curious result, the sum of inverse squares and the solution[24] from an unexpected source of Leonhard Euler's famous *Basel problem.*

Though Euler himself first reached (17.38) by another route, a few routes[25] have since been discovered of which this subsection's is one.

Euler's (17.38) is significant because the series $\sum_{j=1}^{\infty}(1/j^2)$ it sums can arise in contexts that seem to have nothing to do with Fourier analysis or Parseval technique. If the series is summed directly, convergence is slow, but (17.38) gives the exact sum at once—provided that one is alert enough to recognize that (17.38) applies. (Even without leveraging prior knowledge of the value of 2π, using only eqn. 8.43 and Euler's eqn. 17.38, the writer's computer can sum the Basel series to the 0x34-bit—that is, sixteen-decimal place—precision of its `double`-type floating-point representation within 0.2

[24] [182, "Basel problem," 21:42, 30 April 2019]
[25] *Ibid.*

Table 17.1: Sums of inverse squares.

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{1}{j^2} &= \frac{(2\pi)^2}{\mathrm{0x18}} \\ \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} &= \frac{(2\pi)^2}{\mathrm{0x20}} \\ \sum_{j=1}^{\infty} \frac{1}{(2j)^2} &= \frac{(2\pi)^2}{\mathrm{0x60}}\end{aligned}$$

microseconds, whereas to sum the series directly the computer wants more than 0.2 *seconds,* a million times as long.[26] Good mathematical analysis like Euler's achieves such economies.)

In advanced mathematics, apparently unrelated investigations will sometimes deeply connect. The Basel problem affords an example.

Similar operations on the Fourier coefficients (17.23) of the square wave of Fig. 17.1, using $T_1 = 2\pi$ and $A = 1$, yield

$$\sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} = \frac{(2\pi)^2}{\mathrm{0x20}}, \tag{17.39}$$

the sum of inverse odd squares. The sum of inverse even squares, if needed, is given by the difference of (17.38) and (17.39) to be

$$\sum_{j=1}^{\infty} \frac{1}{(2j)^2} = \frac{(2\pi)^2}{\mathrm{0x60}}. \tag{17.40}$$

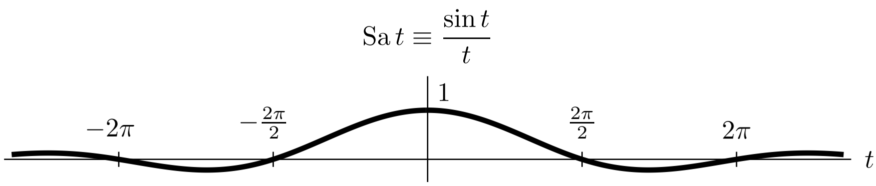
Table 17.1 summarizes.

Incidentally, if you wonder why the table omits sums of plain inverses like $\sum_{j=1}^{\infty}(1/j)$, the reason is that sums of plain inverses diverge. Section 8.10.5 has explained.

See also § 8.10.2 and its Fig. 8.5.

[26]Actually, the direct summation is even worse than this, for floating-point errors will accumulate over millions of terms. Thus, even after taking a million times as long, the direct summation's accuracy is poor. To improve the direct summation's accuracy would require extended-precision floating-point arithmetic, slowing the summation even further. Euler's method, by contrast, can be completed to fine accuracy with a pencil!

Figure 17.10: The sine-argument function.



17.6 The sine-argument function

Equation (17.25) gives the pulse train of Fig. 17.7 its Fourier coefficients, but a better notation for (17.25) is

$$a_j = \eta A \operatorname{Sa} \frac{2\pi\eta j}{2}, \tag{17.41}$$

where

$$\operatorname{Sa} z \equiv \frac{\sin z}{z} \tag{17.42}$$

is the *sine-argument function,*[27] plotted in Fig. 17.10. The function's Taylor series is

$$\operatorname{Sa} z = \sum_{j=0}^{\infty} \prod_{m=1}^{j} \frac{-z^2}{(2m)(2m+1)}, \tag{17.43}$$

the Taylor series of $\sin z$ from Table 8.1, divided by z.

This section introduces the sine-argument function and some of its properties, plus also the related sine integral.[28]

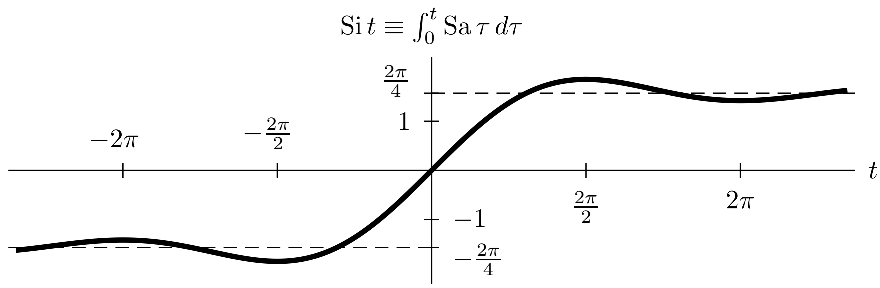
[27]Many (including the author himself in other contexts) call it the *sinc function,* denoting it $\operatorname{sinc}(\cdot)$ and pronouncing it as "sink." Unfortunately, some [129, § 4.3][37, § 2.2][51] use the $\operatorname{sinc}(\cdot)$ notation for another function,

$$\operatorname{sinc}_{\text{alternate}} z \equiv \operatorname{Sa} \frac{2\pi z}{2} = \frac{\sin(2\pi z/2)}{2\pi z/2}.$$

The unambiguous $\operatorname{Sa}(\cdot)$ suits this particular book better, anyway, so this is the notation we will use.

[28]Readers interested in Gibbs' phenomenon, § 17.7, will read the present section because Gibbs depends on its results. Among other readers however some, less interested in special functions than in basic Fourier theory, may find this section unprofitably tedious. They can skip ahead to the start of the next chapter without great loss.

Figure 17.11: The sine integral.



17.6.1 Derivative and integral

The sine-argument function's derivative is computed from the definition (17.42) and the derivative product rule (4.23) to be

$$\frac{d}{dz}\operatorname{Sa} z = \frac{\cos z - \operatorname{Sa} z}{z}. \tag{17.44}$$

The function's integral is expressed as a Taylor series after integrating the function's own Taylor series (17.43) term by term to obtain the form

$$\operatorname{Si} z \equiv \int_0^z \operatorname{Sa}\tau\, d\tau = \sum_{j=0}^{\infty}\left[\frac{z}{2j+1}\prod_{m=1}^{j}\frac{-z^2}{(2m)(2m+1)}\right], \tag{17.45}$$

plotted in Fig. 17.11. Convention gives this integrated function its own name and notation: it calls it the *sine integral*[29,30] and denotes it by $\operatorname{Si}(\cdot)$.

17.6.2 Properties of the sine-argument function

Sine-argument properties include the following.

- The sine-argument function is real over the real domain. That is, if $\Im(t) = 0$ then $\Im(\operatorname{Sa} t) = 0$.
- The zeros of $\operatorname{Sa} z$ occur at $z = n\pi$, $n \neq 0$, $n \in \mathbb{Z}$.

[29] [107, § 3.3]

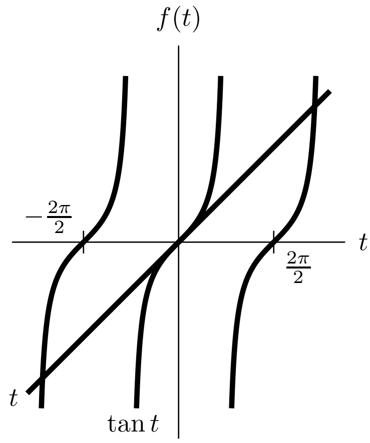
[30] Incidentally, as far as the writer is aware, the name "sine argument" would seem to have been back-constructed from the name "sine integral."

- It is that $|\mathrm{Sa}\, t| < 1$ over the real domain $\Im(t) = 0$ except at the global maximum $t = 0$, where
$$\mathrm{Sa}\, 0 = 1. \tag{17.46}$$
- Over the real domain $\Im(t) = 0$, the function $\mathrm{Sa}\, t$ alternates between distinct positive and negative lobes. Specifically, $(-)^n \mathrm{Sa}(\pm t) > 0$ over $n\pi < t < (n+1)\pi$ for each $n \geq 0$, $n \in \mathbb{Z}$.
- Each of the sine-argument's lobes has but a single peak. That is, over the real domain $\Im(t) = 0$, the derivative $(d/dt)\,\mathrm{Sa}\, t = 0$ is zero at only a single value of t on each lobe.
- The sine-argument function and its derivative converge toward
$$\begin{aligned} \lim_{t\to\pm\infty} \mathrm{Sa}\, t &= 0, \\ \lim_{t\to\pm\infty} \frac{d}{dt}\,\mathrm{Sa}\, t &= 0. \end{aligned} \tag{17.47}$$

Some of these properties are obvious in light of the sine-argument function's definition (17.42). Among the less obvious properties, that $|\mathrm{Sa}\, t| < 1$ says merely that $|\sin t| < |t|$ for nonzero t; which must be true since t, interpreted as an angle—which is to say, as a curved distance about a unit circle—can hardly be shorter than $\sin t$, interpreted as the corresponding direct shortcut to the axis (see Fig. 3.1). For $t = 0$, (8.32) obtains—or, if you prefer, (17.43).

That each of the sine-argument function's lobes should have but a single peak seems right in view of Fig. 17.10 but is nontrivial to prove. To assert that each lobe has but a single peak is to assert that $(d/dt)\,\mathrm{Sa}\, t = 0$ exactly once in each lobe; or, equivalently—after setting the left side of (17.44) to zero, multiplying by $z^2/\cos z$, and changing $t \leftarrow z$—it is to assert that
$$\tan t = t$$
exactly once in each interval
$$n\pi \leq t < (n+1)\pi,\ \ n \geq 0,$$
for $t \geq 0$; and similarly for $t \leq 0$. But according to Table 5.2
$$\frac{d}{dt} \tan t = \frac{1}{\cos^2 t} \geq 1,$$
whereas $dt/dt = 1$, implying that $\tan t$ is everywhere at least as steep as t is—and, well, the most concise way to finish the argument is to draw a picture of it, as in Fig. 17.12, where the curves evidently cannot but intersect exactly once in each interval.

Figure 17.12: The points at which t intersects $\tan t$.



17.6.3 Properties of the sine integral

Properties of the sine integral $\operatorname{Si} t$ of (17.45) include the following.

- Over the real domain $\Im(t) = 0$, the sine integral $\operatorname{Si} t$ is positive for positive t, negative for negative t and, of course, zero for $t = 0$.

- The local extrema of $\operatorname{Si} t$ over the real domain $\Im(t) = 0$ occur at the zeros of $\operatorname{Sa} t$.

- The global maximum and minimum of $\operatorname{Si} t$ over the real domain $\Im(t) = 0$ occur respectively at the first positive and negative zeros of $\operatorname{Sa} t$, which are $t = \pm\pi$.

- The sine integral converges toward

$$\lim_{t\to\pm\infty} \operatorname{Si} t = \pm\frac{2\pi}{4}. \tag{17.48}$$

That the sine integral should reach its local extrema at the sine-argument's zeros ought to be obvious to the extent to which the concept of integration is understood. To explain the other properties it helps first to have expressed the sine integral in the form

$$
\begin{aligned}
\operatorname{Si} t &= S_n + \int_{n\pi}^{t} \operatorname{Sa}\tau\, d\tau, \\
S_n &\equiv \sum_{j=0}^{n-1} U_j, \\
U_j &\equiv \int_{j\pi}^{(j+1)\pi} \operatorname{Sa}\tau\, d\tau, \\
n\pi &\le t < (n+1)\pi, \\
0 &\le n, \quad (j,n) \in \mathbb{Z},
\end{aligned}
$$

where each partial integral U_j integrates over a single lobe of the sine-argument. The several U_j alternate in sign but, because each lobe majorizes the next (§ 8.10.2)—that is, because,[31] in the integrand, $|\operatorname{Sa}\tau| \ge |\operatorname{Sa}(\tau+\pi)|$ for all $\tau \ge 0$—the magnitude of the area under each lobe exceeds that under the next, such that

$$
\begin{aligned}
&0 \le (-)^j \int_{j\pi}^{t} \operatorname{Sa}\tau\, d\tau < (-)^j U_j < (-)^{j-1} U_{j-1}, \\
&j\pi \le t < (j+1)\pi, \\
&0 \le j, \; j \in \mathbb{Z}
\end{aligned}
$$

(except that the U_{j-1} term of the inequality does not apply when $j = 0$, since there is no U_{-1}) and thus that

$$
0 = S_0 < S_{2m} < S_{2m+2} < S_\infty < S_{2m+3} < S_{2m+1} < S_1 \quad \text{for all } m > 0,\ m \in \mathbb{Z};
$$

or in other words that

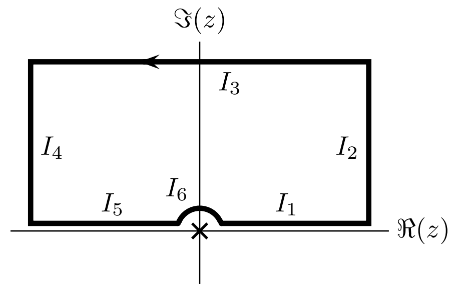
$$
0 = S_0 < S_2 < S_4 < S_6 < S_8 < \cdots < S_\infty < \cdots < S_9 < S_7 < S_5 < S_3 < S_1.
$$

The foregoing applies only when $t \ge 0$ but naturally one can reason similarly for $t \le 0$, concluding that the integral's global maximum and minimum over the real domain occur respectively at the sine-argument function's first positive and negative zeros, $t = \pm\pi$; and further concluding that the integral is positive for all positive t and negative for all negative t.

Equation (17.48) wants some cleverness to calculate and will be the subject of the next subsection.

[31] More rigorously, to give the reason perfectly unambiguously, one could fuss here for a third of a page or so over signs, edges and the like. To do so is left as an exercise to those who aspire to the pure mathematics profession.

Figure 17.13: A complex contour about which to integrate $e^{iz}/i2z$.



17.6.4 The sine integral's limit by complex contour

Equation (17.48) has proposed that the sine integral converges toward a value of $2\pi/4$, but why? The integral's Taylor series (17.45) is impractical to compute for large t and is useless for $t \to \infty$, so it cannot answer the question. To evaluate the integral in the infinite limit, we shall have to think of something cleverer.

Noticing per (5.19) that

$$\operatorname{Sa} z = \frac{e^{+iz} - e^{-iz}}{i2z},$$

rather than trying to integrate the sine-argument function all at once let us first try to integrate just one of its two complex terms, leaving the other term aside to handle later, for the moment computing only

$$I_1 \equiv \int_0^\infty \frac{e^{iz}\,dz}{i2z}.$$

To compute the integral I_1, we will apply the closed-contour technique of § 9.6, choosing a contour in the Argand plane that incorporates I_1 but shuts out the integrand's pole at $z = 0$.

Many contours are possible and one is unlikely to find an amenable contour on the first attempt, but perhaps after several false tries we discover and choose the contour of Fig. 17.13. The integral about the inner semicircle of this contour is

$$I_6 = \int_{C_6} \frac{e^{iz}\,dz}{i2z} = \lim_{\rho\to 0^+} \int_{2\pi/2}^{0} \frac{e^{iz}(i\rho e^{i\phi}\,d\phi)}{i2(\rho e^{i\phi})} = \int_{2\pi/2}^{0} \frac{e^{i0}\,d\phi}{2} = -\frac{2\pi}{4}.$$

The integral across the contour's top segment is

$$I_3 = \int_{C_3} \frac{e^{iz}\,dz}{i2z} = \lim_{a\to\infty} \int_{a}^{-a} \frac{e^{i(x+ia)}\,dx}{i2z} = \lim_{a\to\infty} \int_{-a}^{a} \frac{-e^{ix}e^{-a}\,dx}{i2z},$$

from which, according to the continuous triangle inequality (9.19),

$$|I_3| \le \lim_{a\to\infty} \int_{-a}^{a} \left| \frac{-e^{ix}e^{-a}\,dx}{i2z} \right| = \lim_{a\to\infty} \int_{-a}^{a} \frac{e^{-a}\,dx}{2\,|z|};$$

which, since $0 < a \le |z|$ across the segment, we can weaken to read

$$|I_3| \le \lim_{a\to\infty} \int_{-a}^{a} \frac{e^{-a}\,dx}{2a} = \lim_{a\to\infty} e^{-a} = 0,$$

only possible if

$$I_3 = 0.$$

The integral up the contour's right segment is

$$I_2 = \int_{C_2} \frac{e^{iz}\,dz}{i2z} = \lim_{a\to\infty} \int_{0}^{a} \frac{e^{i(a+iy)}\,dy}{2z} = \lim_{a\to\infty} \int_{0}^{a} \frac{e^{ia}e^{-y}\,dy}{2z},$$

from which, according to the continuous triangle inequality,

$$|I_2| \le \lim_{a\to\infty} \int_{0}^{a} \left| \frac{e^{ia}e^{-y}\,dy}{2z} \right| = \lim_{a\to\infty} \int_{0}^{a} \frac{e^{-y}\,dy}{2\,|z|};$$

which, since $0 < a \le |z|$ across the segment, we can weaken to read

$$|I_2| \le \lim_{a\to\infty} \int_{0}^{a} \frac{e^{-y}\,dy}{2a} = \lim_{a\to\infty} \frac{1}{2a} = 0,$$

only possible if

$$I_2 = 0.$$

The integral down the contour's left segment is

$$I_4 = 0$$

for like reason. Because the contour encloses no pole,

$$\oint \frac{e^{iz}\,dz}{i2z} = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 = 0,$$

which in light of the foregoing calculations implies that

$$I_1 + I_5 = \frac{2\pi}{4}.$$

Now,

$$I_1 = \int_{C_1} \frac{e^{iz}\,dz}{i2z} = \int_0^\infty \frac{e^{ix}\,dx}{i2x}$$

is the integral we wanted to compute in the first place, but what is that I_5? Answer:

$$I_5 = \int_{C_5} \frac{e^{iz}\,dz}{i2z} = \int_{-\infty}^0 \frac{e^{ix}\,dx}{i2x};$$

or, changing $-x \leftarrow x$ by the rule of (9.5) and (9.6),

$$I_5 = \int_0^\infty \frac{-e^{-ix}\,dx}{i2x},$$

which fortuitously happens to integrate the heretofore neglected term of the sine-argument function we started with. Thus,

$$\lim_{t\to\infty} \operatorname{Si} t = \int_0^\infty \operatorname{Sa} x\,dx = \int_0^\infty \frac{e^{+ix} - e^{-ix}}{i2x}\,dx = I_1 + I_5 = \frac{2\pi}{4},$$

which was to be computed.[32]

17.7 Gibbs' phenomenon

Section 17.4.4 has shown how the Fourier series suffices to represent a continuous, repeating waveform. Paradoxically, the chapter's examples have been chiefly of discontinuous waveforms like the square wave. At least in

[32]Integration by closed contour is a subtle technique, is it not? What a finesse this subsection's calculation has been! The author rather strongly sympathizes with the reader who still somehow cannot quite believe that contour integration actually works, but in the case of the sine integral another, quite independent method to evaluate the integral is known and it finds the same number $2\pi/4$. The interested reader can extract this other method from Gibbs' calculation in § 17.7, which refers a sine integral to the known amplitude of a square wave.

We said that it was fortuitous that I_5, which we did not know how to eliminate, turned out to be something we needed anyway; but is it really merely fortuitous, once one has grasped the technique? An integration of $-e^{-iz}/i2z$ is precisely the sort of thing an experienced applied mathematician would expect to fall out as a byproduct of the contour integration of $e^{iz}/i2z$. The trick is to discover the contour from which it actually does fall out, the discovery being a process of informed trial and error.

Fig. 17.2 the Fourier series seems to work for such discontinuous waveforms, though we have never exactly demonstrated that it should work for them, or how. So, what does all this imply?

In one sense, it does not imply much of anything. One can represent a discontinuity by a relatively sharp continuity—as for instance one can represent the Dirac delta of Fig. 7.11 by the triangular pulse of Fig. 17.3, with its sloped edges, if T in (17.12) is sufficiently small—and, considered in this light, the Fourier series works. (See also the nonanalytic pulses of § 17.3 and especially the rolloff pulses of § 17.3.2.) Mathematically however one is more likely to approximate a Fourier series by truncating it after some finite number N of terms; and, indeed, so-called[33] "low-pass" physical systems that naturally suppress high frequencies[34] are common, in which case to truncate the series is more or less the right thing to do. Yet, a significant thing happens when one truncates the Fourier series. *At a discontinuity, the Fourier series oscillates and overshoots.*[35]

Henry Wilbraham investigated this phenomenon as early as 1848. J. Willard Gibbs explored its engineering implications in 1899.[36] Let us along with them refer to the square wave of Fig. 17.2 on page 577. As further Fourier components are added the Fourier waveform better approximates the square wave, but, as we said, it oscillates about and overshoots—it "rings about" in the electrical engineer's vernacular—the square wave's discontinuities (the verb "to ring" here recalling the ringing of a bell or steel beam). This oscillation and overshot turn out to be irreducible, and moreover they can have significant physical effects.

Changing $t - T_1/4 \leftarrow t$ in (17.2) to delay the square wave by a quarter cycle yields that

$$f(t) = \frac{8A}{2\pi} \sum_{j=0}^{\infty} \frac{1}{2j+1} \sin\left[\frac{(2j+1)(2\pi)t}{T_1}\right],$$

which we can, if we like, write as

$$f(t) = \lim_{N\to\infty} \frac{8A}{2\pi} \sum_{j=0}^{N-1} \frac{1}{2j+1} \sin\left[\frac{(2j+1)(2\pi)t}{T_1}\right].$$

[33]So called because they pass low frequencies while suppressing high ones, though systems encountered in practice admittedly typically suffer a middle frequency domain through which frequencies are only partly suppressed.

[34][89, § 15.2]

[35][96]

[36][183][65]

Again changing

$$\Delta v \leftarrow \frac{2(2\pi)t}{T_1}$$

makes this

$$f\left[\frac{T_1}{2(2\pi)}\,\Delta v\right] = \lim_{N\to\infty}\frac{4A}{2\pi}\sum_{j=0}^{N-1}\operatorname{Sa}\left[\left(j+\frac{1}{2}\right)\,\Delta v\right]\,\Delta v.$$

Stipulating that Δv be infinitesimal,

$$0 < \Delta v \ll 1$$

(which in light of the definition of Δv is to stipulate that $0 < t \ll T_1$) such that $dv \equiv \Delta v$ and, therefore, that the summation become an integration; and further defining

$$u \equiv N\,\Delta v;$$

we have that

$$\lim_{N\to\infty} f\left[\frac{T_1}{2(2\pi)N}u\right] = \frac{4A}{2\pi}\int_0^u \operatorname{Sa} v\,dv = \frac{4A}{2\pi}\operatorname{Si} u. \tag{17.49}$$

Equation (17.48) gives us that $\lim_{u\to\infty}\operatorname{Si} u = 2\pi/4$, so (17.49) as it should has it that $f(t) \approx A$ when[37] $t \gtrapprox 0$. When $t \approx 0$ however it gives the waveform locally the sine integral's shape of Fig. 17.11.

Though unexpected the effect can and does actually arise in physical systems. When it does, the maximum value of $f(t)$ is of interest to mechanical and electrical engineers among others because, if an element in an engineered system will overshoot its designed position, the engineer wants to allow safely for the overshot. According to § 17.6.3, the sine integral $\operatorname{Si} u$ reaches its maximum at

$$u = \frac{2\pi}{2},$$

where according to (17.45)[38]

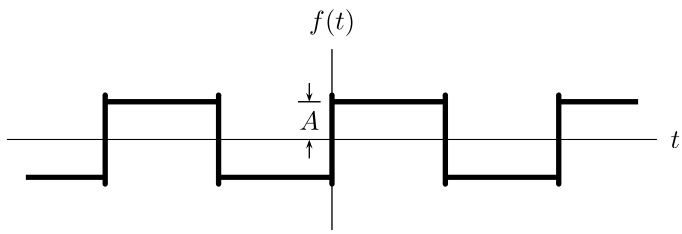
$$\begin{aligned} f_{\max} = \frac{4A}{2\pi}\operatorname{Si}\frac{2\pi}{2} &= \frac{4A}{2\pi}\sum_{j=0}^{\infty}\left[\frac{2\pi/2}{2j+1}\prod_{m=1}^{j}\frac{-(2\pi/2)^2}{(2m)(2m+1)}\right] \approx (\mathrm{0x1.2DD2})A \\ &= (\mathrm{0x1.2DD1\,9DD5\,2786\,71C0\,89EC\,3A50\,CD17\,C577}\ldots)A. \end{aligned}$$

This overshot, peaking momentarily at $(\mathrm{0x1.2DD2})A$, and the associated sine-integral ringing constitute *Gibbs' phenomenon,* as Fig. 17.14 depicts.

[37] Here is an exotic symbol: $\gtrapprox$. It means what it appears to mean, that $t > 0$ and $t \not\approx 0$.

[38] [152, sequence A243267]

Figure 17.14: Gibbs' phenomenon.



We have said that Gibbs' phenomenon is irreducible, and indeed strictly this is so: a true discontinuity, if it is to obey Fourier, must overshoot according to Gibbs. Admittedly as earlier alluded, one can sometimes substantially evade Gibbs by softening a discontinuity's edge as in § 17.3.2, giving the discontinuity a steep but not vertical slope and maybe rounding its corners a little;[39] or, alternately, by rolling the Fourier series off gradually rather than truncating it exactly at N terms. Engineers may do one or the other, or both, explicitly or implicitly, which is why the full Gibbs is not always observed in engineered systems. Nature may do likewise. Neither however is the point. The point is that sharp discontinuities do not behave in the manner one might naïvely have expected, yet that one can still analyze them profitably, adapting this section's subtle technique as the circumstance might demand. A good engineer or other applied mathematician will make himself aware of Gibbs' phenomenon and of the mathematics behind it for this reason.

[39] If the applied mathematician is especially exacting he might represent a discontinuity by the cumulative normal distribution function (20.20) or maybe (if slightly less exacting) by an arctangent, and indeed there are times at which he might do so. However, such extra-fine mathematical craftsmanship is unnecessary to this section's purpose.

Chapter 18

The Fourier transform

The Fourier series of chapter 17 is useful. Applications of the series are extensive. However, the series applies solely to waveforms that repeat—or, at most, to waveforms that can be framed as though they repeated.

An effort to extend the Fourier series to the broader domain of nonrepeating waveforms leads to the *Fourier transform,* this chapter's chief subject.

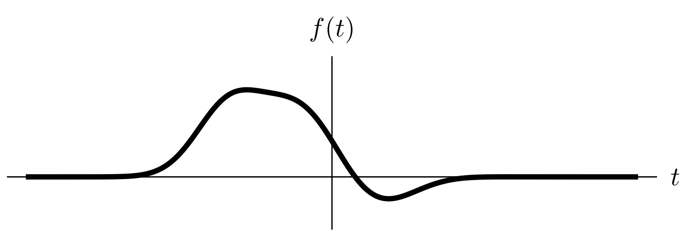
18.1 The Fourier transform

This section extends the Fourier series to derive the Fourier transform.

18.1.1 Fourier's equation

Consider the nonrepeating waveform or *pulse* of Fig. 18.1. Because the

Figure 18.1: A pulse.



pulse does not repeat it has no Fourier series, yet one can however give it something very like a Fourier series in the following way. First, convert the pulse $f(t)$ into the pulse train

$$g(t) \equiv \sum_{n=-\infty}^{\infty} f(t - nT_1),$$

which naturally does repeat,[1] where $T_1 > 0$ is an arbitrary period of repetition whose value you can choose. Second, by (17.22), calculate the Fourier coefficients of this pulse train $g(t)$. Third, use these coefficients in the Fourier series (17.21) to reconstruct

$$g(t) = \sum_{j=-\infty}^{\infty} \left\{ \left[\frac{1}{T_1} \int_{-T_1/2}^{T_1/2} e^{-ij\,\Delta\omega\,\tau} g(\tau)\,d\tau \right] e^{ij\,\Delta\omega\,t} \right\}.$$

Fourth, observing that $\lim_{T_1\to\infty} g(t) = f(t)$, recover from the train the original pulse

$$f(t) = \lim_{T_1\to\infty} \sum_{j=-\infty}^{\infty} \left\{ \left[\frac{1}{T_1} \int_{-T_1/2}^{T_1/2} e^{-ij\,\Delta\omega\,\tau} f(\tau)\,d\tau \right] e^{ij\,\Delta\omega\,t} \right\}$$

[in which we have replaced $g(\tau)$ by $f(\tau)$, supposing that T_1 has grown great enough to separate the several instances $f(\tau - nT_1)$ of which, according to definition, $g(\tau)$ is composed]; or, observing per (17.3) that $\Delta\omega\,T_1 = 2\pi$ and reordering factors,

$$f(t) = \lim_{\Delta\omega\to 0^+} \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} e^{ij\,\Delta\omega\,t} \left[\frac{1}{\sqrt{2\pi}} \int_{-2\pi/2\,\Delta\omega}^{2\pi/2\,\Delta\omega} e^{-ij\,\Delta\omega\,\tau} f(\tau)\,d\tau \right] \Delta\omega.$$

Fifth, defining the symbol $\omega \equiv j\,\Delta\omega$, observe that the summation is really an integration in the limit, such that

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega\tau} f(\tau)\,d\tau \right] d\omega. \tag{18.1}$$

This is *Fourier's equation,* a remarkable, highly significant result.

[1] One could divert rigorously from this point to consider formal requirements against $f(t)$ but for applications it probably suffices that $f(t)$ be limited enough in extent that $g(t)$ exist for all $\Re(T_1) > 0$, $\Im(T_1) = 0$. Formally, such a condition would forbid a function like $f(t) = A\cos\omega_o t$, but one can evade the formality, among other ways, by defining the function as $f(t) = \lim_{T_2\to\infty} \Pi(t/T_2) A\cos\omega_o t$, the $\Pi(\cdot)$ being the rectangular pulse of (17.10). Other pulses of § 17.3 might suit, as well.

18.1.2 The transform and inverse transform

The reader may agree that Fourier's equation (18.1) is curious, but in what way is it remarkable? To answer, let us observe that the quantity within the square brackets of (18.1),

$$F(\omega) \equiv \mathscr{F}\{f(t)\} \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega\tau} f(\tau)\,d\tau, \tag{18.2}$$

is a function not of t but rather of ω. We conventionally give this function the capitalized symbol $F(\omega)$ and name it the *Fourier transform* of $f(t)$, introducing also the notation $\mathscr{F}\{\cdot\}$ (where the script letter $\mathscr{F}$, which stands for "Fourier," is only accidentally the same letter as f and F) as a short form to represent the transformation (18.2) serves to define. Substituting (18.2) into (18.1) and changing $\eta \leftarrow \omega$ as the dummy variable of integration, we have that

$$f(t) = \mathscr{F}^{-1}\{F(\omega)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\eta t} F(\eta)\,d\eta. \tag{18.3}$$

This last is the *inverse Fourier transform* of the function $F(\omega)$.

The Fourier transform (18.2) serves as a continuous measure of a function's frequency content. To understand why this should be so, consider that (18.3) constructs a function $f(t)$ of an infinity of infinitesimally graded complex exponentials and that (18.2) provides the weights $F(\omega)$ for the construction. Indeed, the Fourier transform's complementary equations (18.3) and (18.2) are but continuous versions of the earlier complementary equations (17.21) and (17.22) of the discrete Fourier series. The transform finds even wider application than the series does.[2]

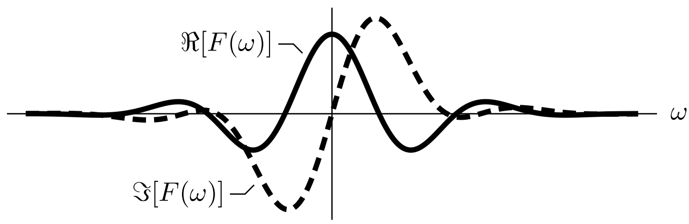
Figure 18.2 plots the Fourier transform of the pulse of Fig. 18.1.

18.1.3 The complementary variables of transformation

If t represents time then ω represents angular frequency as § 17.2 has explained. In this case the function $f(t)$ is said to operate in the *time domain* and the corresponding transformed function $F(\omega)$, in the *frequency domain.* The mutually independent variables ω and t are then the *complementary variables of transformation.*

[2]Regrettably, several alternate definitions and usages of the Fourier series are current. Alternate definitions [129][37] handle the factors of $1/\sqrt{2\pi}$ differently, as for instance in § 19.7. Alternate usages [57] change $-i \leftarrow i$ in certain circumstances. The essential Fourier mathematics however remains the same in any case. The reader can adapt the book's presentation at need to the Fourier definition and usage his colleagues prefer.

Figure 18.2: The Fourier transform of the pulse of Fig. 18.1.



Formally, one can use any two letters in place of the ω and t; and indeed one need not even use two different letters, for it is sometimes easier just to write,

$$\begin{aligned} F(v) = \mathscr{F}\{f(v)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\theta} f(\theta)\, d\theta, \\ f(v) = \mathscr{F}^{-1}\{F(v)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iv\theta} F(\theta)\, d\theta, \\ \mathscr{F} &\equiv \mathscr{F}_{vv}, \end{aligned} \tag{18.4}$$

in which the θ is in itself no variable of transformation but only a dummy variable. To emphasize the distinction between the untransformed and transformed (respectively typically time and frequency) domains, however, scientists and engineers tend to style (18.4) as

$$\begin{aligned} F(\omega) = \mathscr{F}\{f(t)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt, \\ f(t) = \mathscr{F}^{-1}\{F(\omega)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} F(\omega)\, d\omega, \\ \mathscr{F} &\equiv \mathscr{F}_{\omega t}, \end{aligned} \tag{18.5}$$

which are just (18.2) and (18.3) together with their dummy variables changed. For precision of specification, one can affix subscripts as shown: $\mathscr{F}_{vv}$; $\mathscr{F}_{\omega t}$. However, the unadorned $\mathscr{F}$ is normally clear enough in context.)

Whichever letter or letters might be used for the independent variable, the functions

$$f(v) \xrightarrow{\mathscr{F}} F(v) \tag{18.6}$$

constitute a *Fourier transform pair.*

18.1.4 Transforms of the basic nonanalytic pulses

As a Fourier example, consider the triangular pulse $\Lambda(v)$ of (17.10). Its Fourier transform according to (18.4) is

$$\begin{aligned}\mathscr{F}\{\Lambda(v)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iv\theta}\Lambda(\theta)\,d\theta \\ &= \frac{1}{\sqrt{2\pi}}\left\{\int_{-1}^{0} e^{-iv\theta}(1+\theta)\,d\theta + \int_{0}^{1} e^{-iv\theta}(1-\theta)\,d\theta\right\}.\end{aligned}$$

Evaluating the integrals according to Table 9.1's antiderivative that $\theta e^{-iv\theta} = [d/d\theta][e^{-iv\theta}(1+iv\theta)/v^2]$,

$$\begin{aligned}\mathscr{F}\{\Lambda(v)\} &= \frac{1}{v^2\sqrt{2\pi}}\left\{\left[e^{-iv\theta}[1+(iv)(1+\theta)]\right]_{\theta=-1}^{0}\right. \\ &\qquad\qquad \left. + \left[e^{-iv\theta}[-1+(iv)(1-\theta)]\right]_{\theta=0}^{1}\right\} \\ &= \frac{\operatorname{Sa}^2(v/2)}{\sqrt{2\pi}},\end{aligned}$$

where $\operatorname{Sa}(\cdot)$ is the sine-argument function of (17.42). Thus we find the Fourier transform pair

$$\Lambda(v) \stackrel{\mathscr{F}}{\to} \frac{\operatorname{Sa}^2(v/2)}{\sqrt{2\pi}}. \tag{18.7}$$

One can compute other Fourier transform pairs in like manner, such as[3]

$$\Pi(v) \stackrel{\mathscr{F}}{\to} \frac{\operatorname{Sa}(v/2)}{\sqrt{2\pi}}, \tag{18.8}$$

$$\Psi(v) \stackrel{\mathscr{F}}{\to} \frac{\operatorname{Sa} v}{\sqrt{2\pi}\,[1-(v/\pi)^2]}. \tag{18.9}$$

One can compute yet further transform pairs by the duality rule and other properties of § 18.2.

The algebra can be intricate, though. As a second Fourier example,

[3]To verify (18.8) and (18.9) is left as an exercise. Hint toward (18.9): $\sin(v \pm \pi) = -\sin v$.

consider the raised cosine-rolloff pulse $\Psi_r(t)$ of (17.17):

$$\begin{aligned}
&\left(\sqrt{2\pi}\right) \mathscr{F}\left\{\Psi_r(v)\right\} \\
&= \int_{-(1+r)/2}^{-(1-r)/2} \left\{ \frac{e^{-iv\theta}}{2} + \frac{e^{i[(-v+\pi/r)\theta+2\pi/4r]}}{i4} - \frac{e^{i[(-v-\pi/r)\theta-2\pi/4r]}}{i4} \right\} d\theta \\
&\quad + \int_{-(1-r)/2}^{(1-r)/2} e^{-iv\theta}\, d\theta \\
&\quad + \int_{(1-r)/2}^{(1+r)/2} \left\{ \frac{e^{-iv\theta}}{2} + \frac{e^{i[(-v-\pi/r)\theta+2\pi/4r]}}{i4} - \frac{e^{i[(-v+\pi/r)\theta-2\pi/4r]}}{i4} \right\} d\theta \\
&= \left\{ -\frac{e^{-iv\theta}}{i2v} - \frac{e^{i[(-v+\pi/r)\theta+2\pi/4r]}}{4(-v+\pi/r)} + \frac{e^{i[(-v-\pi/r)\theta-2\pi/4r]}}{4(-v-\pi/r)} \right\}_{-(1+r)/2}^{-(1-r)/2} \\
&\quad + \left\{ -\frac{e^{-iv\theta}}{iv} \right\}_{-(1-r)/2}^{(1-r)/2} \\
&\quad + \left\{ -\frac{e^{-iv\theta}}{i2v} - \frac{e^{i[(-v-\pi/r)\theta+2\pi/4r]}}{4(-v-\pi/r)} + \frac{e^{i[(-v+\pi/r)\theta-2\pi/4r]}}{4(-v+\pi/r)} \right\}_{(1-r)/2}^{(1+r)/2},
\end{aligned}$$

and so on. After two more pages or so of algebra (left as a tedious though not especially difficult exercise to try the accuracy of the interested reader's pencil), the result is that[4]

$$\Psi_r(v) \xrightarrow{\mathscr{F}} \frac{\cos(rv/2)\,\mathrm{Sa}(v/2)}{\sqrt{2\pi}\,[1-(rv/\pi)^2]}. \tag{18.10}$$

Unless $r = 0$, the v^2 in the transform's denominator lends $\mathscr{F}\{\Psi_r(v)\}$ the possibly significant property that $\int_{-\infty}^{\infty} |\mathscr{F}\{\Psi_r(v)\}|\, dv$ converges as $\int dv/v^3$ does, whereas $\int_{-\infty}^{\infty} |\mathscr{F}\{\Pi(v)\}|\, dv$ does not converge but diverges as $\int dv/v$. Using a small but nonzero r, applications (§ 18.1.5) sometimes replace $\Pi(v)$ by $\Psi_r(v)$ [or, alternately, if convergence[5] less aggressive than that of $\Psi_r(v)$ is tolerable, maybe instead by the simpler $\Lambda_r(v)$] for this or related reasons.

[4]The conscientious reader might check (18.10) against (18.8) and (18.9), the former with $r = 0$, the latter with $r = 1$. The former looks all right but the latter looks wrong at first glance, until one recalls from Table 3.3 the trigonometric identity that $\sin 2\alpha = 2 \sin\alpha \cos\alpha$.

[5]Rather than "convergence," an electrical engineer might instead say "band confinement," where by "band" the engineer means "range of frequencies." The engineer would change $t \leftarrow v$, transform via $\mathscr{F}_{\omega t}$ rather than $\mathscr{F}_{vv}$, and, in the result, observe that $\Psi_r(t)$ is even less active at high values of the frequency ω than $\Lambda_r(t)$ is.

Similarly,[6]

$$\Lambda_r(v) \xrightarrow{\mathscr{F}} \frac{\operatorname{Sa}(rv/2)\operatorname{Sa}(v/2)}{\sqrt{2\pi}}. \tag{18.11}$$

One thing left to consider is the values of the raised-cosine transforms when the denominators of (18.9) and (18.10) vanish. For the latter,

$$\begin{aligned}
\lim_{rv\to\pm\pi} \frac{\cos(rv/2)\operatorname{Sa}(v/2)}{\sqrt{2\pi}\,[1-(rv/\pi)^2]} &= \lim_{rv\mp\pi\to 0} \frac{\sin[(\pi\mp rv)/2]\operatorname{Sa}(v/2)}{\sqrt{2\pi}(1\mp rv/\pi)(1\pm rv/\pi)} \\
&= \lim_{rv\mp\pi\to 0} \frac{\pi^2\sin[(\pi\mp rv)/2]\operatorname{Sa}(v/2)}{\sqrt{2\pi}(\pi\mp rv)(\pi\pm rv)} \\
&= \lim_{rv\mp\pi\to 0} \frac{(2\pi)^{3/2}\operatorname{Sa}[(\pi\mp rv)/2]\operatorname{Sa}(v/2)}{8(\pi\pm rv)} \\
&= \frac{\left(\sqrt{2\pi}\right)\operatorname{Sa}(v/2)}{8} = \frac{\left(\sqrt{2\pi}\right)\operatorname{Sa}(2\pi/4r)}{8}.
\end{aligned}$$

For the former, using l'Hôpital's rule (4.29) and the derivative (17.44),

$$\lim_{v\to\pm\pi} \frac{\operatorname{Sa} v}{\sqrt{2\pi}\,[1-(v/\pi)^2]} = \left.\frac{\operatorname{Sa} v - \cos v}{2\sqrt{2\pi}(v/\pi)^2}\right|_{v=\pm\pi} = \frac{1}{2\sqrt{2\pi}},$$

which, reassuringly, agrees with the former at $r = 1$. Summarizing,

$$\begin{aligned}
\mathscr{F}\{\Psi(v)\}_{v=\pm\pi} &= \frac{1}{2\sqrt{2\pi}}, \\
\mathscr{F}\{\Psi_r(v)\}_{rv=\pm\pi} &= \frac{\left(\sqrt{2\pi}\right)\operatorname{Sa}(v/2)}{8} = \frac{\left(\sqrt{2\pi}\right)\operatorname{Sa}(2\pi/4r)}{8}.
\end{aligned} \tag{18.12}$$

18.1.5 Approximation to an arbitrary pulse

Before closing the section, we should briefly notice an important application of the several nonanalytic pulses of § 17.3, whose Fourier transforms § 18.1.4 has computed and Table 18.4 will list. Sampling a pulse $f(t)$ at equal, finite intervals of Δt, one can approximate the pulse as

$$\tilde{f}(t) \equiv \sum_{k=-\infty}^{\infty} f(k\,\Delta t)h\left(\frac{t-k\,\Delta t}{\Delta t}\right), \tag{18.13}$$

[6] If you work these out for yourself and get

$$\Lambda_r(v) \xrightarrow{\mathscr{F}} \frac{2}{\left(\sqrt{2\pi}\right)rv^2}\left[\cos\frac{(1-r)v}{2} - \cos\frac{(1+r)v}{2}\right]$$

then your answer is right. Apply a sum-of-angles identity of Table 3.3 to reach the formula the narrative gives.

where $h(t)$ is any of the basic nonanalytic pulses of § 17.3.1 and Fig. 17.3, whether $\Pi(t)$, $\Lambda(t)$ or $\Psi(t)$; or is either of the rolloff variants of § 17.3.2 and Fig. 17.4, whether $\Lambda_r(t)$ or $\Psi_r(t)$. Naturally, there are constraints for this to work. For instance, $f(t)$ must not change so quickly or abruptly that sufficiently dense samples fail to approximate the function. Fortunately, whether $f(t)$ submits to such constraints tends to be clearer in an actual application with a specific, concrete $f(t)$ than it is in the abstract; so, other than to warn the reader regarding any $f(t)$ that suffers an infinite slope—like $f(t) = (te^{-|t|^2/2})/\sqrt{|t|}$, for example—we shall not further study the constraints here (but see § 19.2, which develops constructive functions that impose fewer constraints).

Having approximated the pulse by (18.13), having thus reduced the pulse to a superposition $\tilde{f}(t)$ of pulses whose Fourier transforms we already know, we can approximate that $\mathscr{F}\{f(t)\} \approx \mathscr{F}\{\tilde{f}(t)\}$, where

$$\mathscr{F}\left\{\tilde{f}(t)\right\} = \sum_{k=-\infty}^{\infty} f(k\,\Delta t)\mathscr{F}\left\{h\left(\frac{t-k\,\Delta t}{\Delta t}\right)\right\}. \tag{18.14}$$

Qualifications regarding convergence and such could be multiplied against (18.14) but, again, that is not our purpose here. Other than to note that the square pulse $\Pi(t)$ might, in some applications, be a poor choice for $h(t)$ because the absolute integral $\int_{-\infty}^{\infty} |\mathscr{F}\{\Pi(t)\}|\,d\omega$ of its transform fails to converge, we will leave the question in that form.

We said that we already knew the transforms of the various basic nonanalytic pulses. However, (18.13) and (18.14) have used $h[(t-k\,\Delta t)/\Delta t]$ rather than a plain $h(t)$. Fortunately, the conversion is not difficult. It is that

$$\mathscr{F}\left\{h\left(\frac{t-k\,\Delta t}{\Delta t}\right)\right\} = e^{-ik\,\Delta t\,\omega}\,\Delta t\,H(\omega\,\Delta t), \tag{18.15}$$

as we shall see in Table 18.1 of § 18.2, next.

One point in closing: to sample $f(t)$ at a single instant $t = k\,\Delta t$ and then to disregard the detailed evolution of $f(t)$ until the next sampling instant $t = (k+1)\,\Delta t$ can be less than ideal when the sampling interval Δt is less fine than one would like. In this case, if practicable, one might prefer to change

$$\int_{-\infty}^{\infty} h_{\mathrm{sampling}}(\tau) f(k\,\Delta t - \tau)\,d\tau \leftarrow f(k\,\Delta t) \tag{18.16}$$

in (18.13) and (18.14) to smear out the sample a bit—and indeed an engineer might be obliged to do so in any case due to the electromechanical limitations of his sampling device. The $h_{\mathrm{sampling}}(\tau)$ is called a *sampling window* and it can be any of the basic nonanalytic pulses $h(t)$ can be, among others. It may, but need not be, the same pulse $h(t)$ is.[7]

18.2 Properties of the Fourier transform

The Fourier transform obeys an algebra of its own, exhibiting properties the mathematician can exploit to extend the transform's reach. This section derives and lists several such properties.

18.2.1 Symmetries of the real and imaginary parts

In Fig. 18.2, page 618, each of the real and imaginary parts of the Fourier transform is symmetrical (or at least each *looks* symmetrical), though the imaginary symmetry differs from the real. The present subsection analytically develops the symmetries.

The Fourier transform of a function's conjugate according to (18.4) is

$$\mathscr{F}\{f^*(v)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\theta} f^*(\theta)\, d\theta = \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iv^*\theta} f(\theta)\, d\theta \right]^*,$$

in which we have taken advantage of the fact that the integrand's dummy variable $\theta = \theta^*$ happens to be real. On the other hand, within the paragraph's first equation just above,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iv^*\theta} f(\theta)\, d\theta = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(-v^*)\theta} f(\theta)\, d\theta = F(-v^*),$$

implying that[8]

$$\begin{aligned} f^*(v) &\xrightarrow{\mathscr{F}} F^*(-v^*); \\ f^*(t) &\xrightarrow{\mathscr{F}} F^*(-\omega^*). \end{aligned} \tag{18.17}$$

[7][44][140, § 5.6.3]

[8]From past experience with complex conjugation, an applied mathematician might naturally have expected of (18.17) that $f^*(v) \xrightarrow{\mathscr{F}} F^*(v)$, but the expectation though natural would have been incorrect. Unlike most of the book's mathematics before chapter 17, eqns. (17.21) and (17.22)—and thus ultimately also the Fourier transform's definition (18.2) or (18.4)—have arbitrarily chosen a particular sign for the i in the phasing factor $e^{-ij\,\Delta\omega\,\tau}$ or $e^{-iv\theta}$, which phasing factor the Fourier integration bakes into the transformed function $F(v)$, so to speak. The Fourier transform as such therefore does not meet § 2.11.2's condition for (2.78) to hold. Fortunately, (18.17) does hold.

Viewed from another angle, it must be so, because Fourier transforms real functions into complex ones. See Figs. 18.1 and 18.2.

If we express the real and imaginary parts of $f(v)$ in the style of (2.72) as

$$\Re[f(v)] = \frac{f(v) + f^*(v)}{2},$$
$$\Im[f(v)] = \frac{f(v) - f^*(v)}{i2},$$

then the Fourier transforms of these parts according to (18.17) are[9]

$$\begin{aligned}\Re[f(v)] &\stackrel{\mathscr{F}}{\to} \frac{F(v) + F^*(-v^*)}{2},\\ \Im[f(v)] &\stackrel{\mathscr{F}}{\to} \frac{F(v) - F^*(-v^*)}{i2}.\end{aligned} \tag{18.18}$$

For real v and an $f(v)$ which is itself real for all real v, the latter line becomes

$$0 \stackrel{\mathscr{F}}{\to} \frac{F(v) - F^*(-v)}{i2} \quad \text{if } \Im(v) = 0 \text{ and, for all such } v,\ \Im[f(v)] = 0,$$

whereby[10]

$$F(v) = F^*(-v) \quad \text{if } \Im(v) = 0 \text{ and, for all such } v,\ \Im[f(v)] = 0. \tag{18.19}$$

Interpreted, (18.19) says for real v and $f(v)$ that the plot of $\Re[F(v)]$ must be symmetric about the vertical axis whereas the plot of $\Im[F(v)]$ must be symmetric about the origin, as Fig. 18.2 has illustrated.

18.2.2 Duality

Changing $-v \leftarrow v$ makes (18.4)'s second line to read

$$f(-v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\theta} F(\theta)\, d\theta.$$

However, according to (18.4)'s first line, this says neither more nor less than that

$$\begin{aligned}F(v) &\stackrel{\mathscr{F}}{\to} f(-v),\\ F(t) &\stackrel{\mathscr{F}}{\to} f(-\omega),\end{aligned} \tag{18.20}$$

[9]The precisely orderly reader might note that a forward reference to Table 18.1 is here implied; but the property referred to, Fourier superposition $A_1 f_1(v) + A_2 f_2(v) \stackrel{\mathscr{F}}{\to} A_1 F_1(v) + A_2 F_2(v)$, which does not depend on this subsection's results anyway, is so trivial to prove that we will not bother about the precise ordering in this instance.

[10]A professional mathematician might object that we had never established a one-to-one correspondence between a function and its transform. On the other hand, recognizing the applied spirit of the present work, the professional might waive the objection—if not with pleasure, then at least with a practical degree of indulgence—but see also § 17.4.4.

which is that the transform of the transform is the original function with the independent variable reversed, an interesting and useful property. It is entertaining, and moreover enlightening, to combine (18.6) and (18.20) to form the endless transform progression

$$\cdots \stackrel{\mathscr{F}}{\to} f(v) \stackrel{\mathscr{F}}{\to} F(v) \stackrel{\mathscr{F}}{\to} f(-v) \stackrel{\mathscr{F}}{\to} F(-v) \stackrel{\mathscr{F}}{\to} f(v) \stackrel{\mathscr{F}}{\to} \cdots \tag{18.21}$$

Equation (18.20) or alternately (18.21) expresses the Fourier transform's property of *duality.*

For an example of duality, recall that § 18.1.4 has computed the transform pairs

$$\begin{aligned} \Pi(v) &\stackrel{\mathscr{F}}{\to} \frac{\operatorname{Sa}(v/2)}{\sqrt{2\pi}}, \\ \Lambda(v) &\stackrel{\mathscr{F}}{\to} \frac{\operatorname{Sa}^2(v/2)}{\sqrt{2\pi}}, \\ \Psi(v) &\stackrel{\mathscr{F}}{\to} \frac{\operatorname{Sa} v}{\sqrt{2\pi}\,[1-(v/\pi)^2]}. \end{aligned}$$

Application of (18.20) yields the additional, dual transform pairs[11]

$$\begin{aligned} \frac{\operatorname{Sa}(v/2)}{\sqrt{2\pi}} &\stackrel{\mathscr{F}}{\to} \Pi(v), \\ \frac{\operatorname{Sa}^2(v/2)}{\sqrt{2\pi}} &\stackrel{\mathscr{F}}{\to} \Lambda(v), \\ \frac{\operatorname{Sa} v}{\sqrt{2\pi}\,[1-(v/\pi)^2]} &\stackrel{\mathscr{F}}{\to} \Psi(v), \end{aligned} \tag{18.22}$$

in which that $\Pi(-v) = \Pi(v)$, $\Lambda(-v) = \Lambda(v)$ and $\Psi(-v) = \Psi(v)$ are observed (but eqn. 18.20 works as well on pulses that lack such symmetry). Without duality, to use (18.4) to compute the transform pairs of (18.22) might have been hard, but with duality it's pretty easy, as you see. (Section 18.3 incidentally will further improve eqn. 18.22.)

18.2.3 The Fourier transform of the Dirac delta

Section 18.3 will compute several Fourier transform pairs but one particular pair is so significant, so illustrative, so curious, and so easy to compute that

[11]The last of these three duals though logical is admittedly, probably not very useful—except maybe in a few special cases in which the applied mathematician, already expecting it, will hardly need to look it up. Table 18.4 omits it.

we will pause to compute it and its dual now. Applying (18.4) to the Dirac delta (7.23) and invoking its sifting property (7.25), we find that

$$\delta(v) \xrightarrow{\mathscr{F}} \frac{1}{\sqrt{2\pi}}, \tag{18.23}$$

the dual of which according to (18.21) is

$$1 \xrightarrow{\mathscr{F}} \left(\sqrt{2\pi}\right) \delta(v) \tag{18.24}$$

inasmuch as $\delta(-v) = \delta(v)$.

The duality rule again incidentally proves its worth in (18.24). Had we tried to calculate the Fourier transform of 1—that is, of $f(v) \equiv 1$—directly according to (18.4) we would have found the pair $1 \xrightarrow{\mathscr{F}} (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} e^{-iv\theta}\,d\theta$, the right side of which features an integral impossible to evaluate. A limit of some kind might perhaps have been enforced to circumvent the impossibility, but as again you see, duality is easier.

18.2.4 Delay and shift

Applying (18.4) to $f(v-a)$ and changing $\xi \leftarrow \theta - a$, we have the transform property of *delay:*

$$f(v-a) \xrightarrow{\mathscr{F}} e^{-iav} F(v). \tag{18.25}$$

Applying (18.4) to $e^{iav} f(v)$, we have the transform property of *frequency shift:*

$$e^{iav} f(v) \xrightarrow{\mathscr{F}} F(v-a). \tag{18.26}$$

18.2.5 Metaduality

Section 18.2.2 has shown how to compose the dual of a Fourier transform pair. One can likewise compose the *metadual*[12] of a Fourier transform *property,* but to do so correctly wants delicate handling.

Defining[13]

$$\begin{aligned} F(w) &\equiv \mathscr{F}\{f(u)\}, \\ \Phi(u) &\equiv \mathscr{F}\{\phi(w)\}, \\ \phi(w) &\equiv F(w), \end{aligned}$$

[12]Other books the writer has read just call it a "dual." However, the book you are reading finds it expedient to disambiguate by modifying the term where the term is used of properties rather than of mere pairs.

[13]The Roman w of this subsection is not the Greek ω of § 18.1.1.

one can write (18.21) as

$$\begin{array}{cccccccccccc} \cdots & \xrightarrow{\mathscr{F}} & \Phi(-u) & \xrightarrow{\mathscr{F}} & \phi(w) & \xrightarrow{\mathscr{F}} & \Phi(u) & \xrightarrow{\mathscr{F}} & \phi(-w) & \xrightarrow{\mathscr{F}} & \cdots \\ \cdots & \xrightarrow{\mathscr{F}} & f(u) & \xrightarrow{\mathscr{F}} & F(w) & \xrightarrow{\mathscr{F}} & f(-u) & \xrightarrow{\mathscr{F}} & F(-w) & \xrightarrow{\mathscr{F}} & \cdots \end{array} \tag{18.27}$$

in which $\phi(w) = F(w)$ are vertically aligned, $\Phi(u) = f(-u)$ are vertically aligned, and so on. Similarly,

$$\begin{array}{cccccccccccc} \cdots & \xrightarrow{\mathscr{F}} & \Gamma(-v) & \xrightarrow{\mathscr{F}} & \gamma(v) & \xrightarrow{\mathscr{F}} & \Gamma(v) & \xrightarrow{\mathscr{F}} & \gamma(-v) & \xrightarrow{\mathscr{F}} & \cdots \\ \cdots & \xrightarrow{\mathscr{F}} & g(v) & \xrightarrow{\mathscr{F}} & G(v) & \xrightarrow{\mathscr{F}} & g(-v) & \xrightarrow{\mathscr{F}} & G(-v) & \xrightarrow{\mathscr{F}} & \cdots \end{array} \tag{18.28}$$

These are just to repeat (18.21) in various symbols $\phi \leftarrow F$ and $\gamma \leftarrow G$, so they say nothing new, but the variant symbology helps for example as follows.

Let the delay property (18.25) be styled as

$$\gamma(v) \xrightarrow{\mathscr{F}} \Gamma(v),$$

where

$$\begin{aligned} \gamma(v) &\equiv \phi(v-a), \\ \Gamma(v) &= e^{-iav}\Phi(v), \\ \gamma(-v) &= \phi(-v-a), \\ \Gamma(-v) &= e^{iav}\Phi(-v), \\ w &\equiv v-a, \\ u &\equiv v, \end{aligned}$$

the w and u defined such that the right sides of the display's first and second lines respectively mention $\phi(w)$—that is, $\phi(v-a)$—and $\Phi(u)$—that is, $\Phi(v)$; whereas the right sides of the display's third and fourth lines do not purposely mention $\phi(-w)$ or $\Phi(-u)$ (although they might mention the one or the other by accident). Indeed, the third and fourth lines are written merely by substituting $-v \leftarrow v$ everywhere into the first and second.

Having written the four lines (for $\gamma[v]$, $\Gamma[v]$, $\gamma[-v]$ and $\Gamma[-v]$), we change symbols downward a row with respect to each of (18.27) and (18.28), revealing the pair

$$G(v) \xrightarrow{\mathscr{F}} g(-v),$$

where

$$\begin{aligned} G(v) &= F(v-a), \\ g(-v) &= e^{-iav}f(-v), \\ G(-v) &= F(-v-a), \\ g(v) &= e^{iav}f(v). \end{aligned}$$

Here is where the delicate handling the subsection's leading paragraph has mentioned is required.

1. In the display's first line, $G(v) \leftarrow \gamma(v)$ is changed on the left according to (18.28) and $F(w) \leftarrow \phi(w)$ on the right according to (18.27).

2. In the display's second line, $g(-v) \leftarrow \Gamma(v)$ is changed on the left according to (18.28) and $f(-u) \leftarrow \Phi(u)$ on the right according to (18.27). The latter change is tricky because u is defined only for use between the parentheses of $f(\cdot)$ or $\Phi(\cdot)$ so, despite that $u \equiv v$ in the present example, the second line's factor e^{-iav} *outside* the parentheses remains unchanged.

3. In the display's third line, an easier rule is used without direct reference to the last paragraph's display: the third line is the first line of *this* paragraph's display with the change $-v \leftarrow v$.

4. In the display's fourth line, the easier rule is used again: the fourth line is the second line with the change $-v \leftarrow v$.

Observe that the changes in all four lines are merely changes of symbol: the changes do not alter the meaning of any of the four lines. The changes in the first and fourth lines do however make it plainer that, since $g(v) \xrightarrow{\mathscr{F}} G(v)$,

$$e^{iav}f(v) \xrightarrow{\mathscr{F}} F(v-a),$$

which is (18.26). Apparently, the frequency-shifting property is the metadual of the delay property.

As an exercise to check understanding, try now the reverse problem with your own pencil: derive the delay property as the metadual of the frequency-shifting property rather than the other way around. Hints: the solution to the reverse problem begins $\gamma(v) \equiv e^{iav}\phi(v)$, $\Gamma(v) = \Phi(v-a)$; in the reverse problem, $u \not\equiv v$. Notice incidentally that the letter v does not appear outside the parentheses on the second line's right side in the reverse problem but, if it did, the reverse problem would handle it as the forward problem has done.[14]

Besides (18.27) and (18.28), the reverse skew is just as possible:

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{\mathscr{F}} & \phi(w) & \xrightarrow{\mathscr{F}} & \Phi(u) & \xrightarrow{\mathscr{F}} & \phi(-w) & \xrightarrow{\mathscr{F}} & \Phi(-u) & \xrightarrow{\mathscr{F}} & \cdots \\ \cdots & \xrightarrow{\mathscr{F}} & F(-w) & \xrightarrow{\mathscr{F}} & f(u) & \xrightarrow{\mathscr{F}} & F(w) & \xrightarrow{\mathscr{F}} & f(-u) & \xrightarrow{\mathscr{F}} & \cdots \end{array}$$

and

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{\mathscr{F}} & \gamma(v) & \xrightarrow{\mathscr{F}} & \Gamma(v) & \xrightarrow{\mathscr{F}} & \gamma(-v) & \xrightarrow{\mathscr{F}} & \Gamma(-v) & \xrightarrow{\mathscr{F}} & \cdots \\ \cdots & \xrightarrow{\mathscr{F}} & G(-v) & \xrightarrow{\mathscr{F}} & g(v) & \xrightarrow{\mathscr{F}} & G(v) & \xrightarrow{\mathscr{F}} & g(-v) & \xrightarrow{\mathscr{F}} & \cdots \end{array}$$

though the forward skew of (18.27) and (18.28) probably suffices. Either way, a lot of letters are used—$\phi\Phi fF(wu)$ and $\gamma\Gamma gG(v)$ [and you can use yet more letters like $\chi\mathrm{X}hH(w_\chi u_\mathrm{X})$ if you have an extra function to transform as, for instance, while deriving eqn. 18.34]—but the letters serve to keep the various relations straight and, anyway, you don't need so many letters to compute the dual (18.20) of an ordinary transform pair but only to compute the metadual of a transform property.

18.2.6 Summary of properties

Table 18.1 summarizes properties of the Fourier transform.

The table's first three properties have been proved earlier in this section. The table's fourth property, that

$$Af(\alpha v) \xrightarrow{\mathscr{F}} \frac{A}{|\alpha|}F\left(\frac{v}{\alpha}\right) \quad \text{if } \Im(\alpha) = 0,\ \Re(\alpha) \neq 0, \tag{18.29}$$

which is the *scaling property* of the Fourier transform, is proved by applying (18.4) to $Af(\alpha v)$ and then changing $\xi \leftarrow \alpha\theta$ (the magnitude sign $|\cdot|$ coming because α, if negative, reverses Fourier's infinite limits of integration in eqn. 18.4; see § 9.3 and its eqn. 9.10). The table's fifth property is merely the fourth with an alternate scale. The table's sixth property applies the fourth and then the second, in that sequence. The table's seventh property is proved trivially.

[14]One can compose a more complicated metadual problem in which $u \neq v$ *and* the v appears outside the parentheses by changing letters $j \leftarrow f$, $J \leftarrow F$ and $b \leftarrow a$ in (18.25) and applying the result as a property to (18.26) as a pair, or alternately by changing letters $j \leftarrow f$, $J \leftarrow F$ and $b \leftarrow a$ in (18.26) and applying the result as a property to (18.25) as a pair. The composition and subsequent solution of these two problems is recommended to the interested reader as a supplementary exercise.

Table 18.1: Properties of the Fourier transform.

$$
\begin{aligned}
F(v) &= F^*(-v) \ \text{ if } \Im(v) = 0 \text{ and,} \\
& \qquad \text{for all such } v,\ \Im[f(v)] = 0. \\
f(v-a) &\stackrel{\mathscr{F}}{\to} e^{-iav} F(v) \\
e^{iav} f(v) &\stackrel{\mathscr{F}}{\to} F(v-a) \\
A f(\alpha v) &\stackrel{\mathscr{F}}{\to} \frac{A}{|\alpha|} F\left(\frac{v}{\alpha}\right) \\
\left(\sqrt{|\alpha|}\right) A f(\alpha v) &\stackrel{\mathscr{F}}{\to} \frac{A}{\sqrt{|\alpha|}} F\left(\frac{v}{\alpha}\right) \\
A f[(\alpha)(v-a)] &\stackrel{\mathscr{F}}{\to} \frac{A e^{-iav}}{|\alpha|} F\left(\frac{v}{\alpha}\right) \\
& \qquad \Im(\alpha) = 0,\ \Re(\alpha) \neq 0 \\
A_1 f_1(v) + A_2 f_2(v) &\stackrel{\mathscr{F}}{\to} A_1 F_1(v) + A_2 F_2(v) \\
\frac{d}{dv} f(v) &\stackrel{\mathscr{F}}{\to} iv F(v) \\
-iv f(v) &\stackrel{\mathscr{F}}{\to} \frac{d}{dv} F(v) \\
\frac{d^n}{dv^n} f(v) &\stackrel{\mathscr{F}}{\to} (iv)^n F(v) \\
(-iv)^n f(v) &\stackrel{\mathscr{F}}{\to} \frac{d^n}{dv^n} F(v) \\
& \qquad n \in \mathbb{Z},\ n \geq 0 \\
\int_{-\infty}^{v} f(\tau)\, d\tau &\stackrel{\mathscr{F}}{\to} \frac{F(v)}{iv} + \pi F(0) \delta(v) \\
\int_{-\infty}^{\infty} h^*(v) f(v)\, dv &= \int_{-\infty}^{\infty} H^*(v) F(v)\, dv \\
\int_{-\infty}^{\infty} |f(v)|^2\, dv &= \int_{-\infty}^{\infty} |F(v)|^2\, dv \\
a_j &= \frac{\Delta\omega}{\sqrt{2\pi}} F(j\,\Delta\omega)
\end{aligned}
$$

The table's eighth through eleventh properties begin from the derivative of the inverse Fourier transform; that is, of (18.4)'s second line. This derivative is

$$\frac{d}{dv}f(v) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iv\theta}[i\theta F(\theta)]\,d\theta = \mathscr{F}^{-1}\{ivF(v)\},$$

which implies that

$$\mathscr{F}\left\{\frac{d}{dv}f(v)\right\} = ivF(v),$$

the table's eighth property. The metadual (§ 18.2.5) of the eighth property is the table's ninth property, during the computation of which[15] one observes that,

$$\text{if } \gamma(v) = \left[\frac{d}{dr}\phi(r)\right]_{r=v},\ \text{then } \gamma(-v) = -\left[\frac{d}{dr}\phi(r)\right]_{r=-v}, \tag{18.30}$$

a fact whose truth can be demonstrated via (4.13)'s definition of the derivative or, easier, can be seen by sketching on a sheet of paper some arbitrary, asymmetric function (like, say, $\phi[v] \equiv e^{v/3}$) and a visual approximation to its derivative. The table's tenth and eleventh properties come by repeated application of the eighth and ninth.

The table's twelfth property is (18.60). Section 18.5 will derive it.

The table's final three properties are (18.45), (18.46) and (19.4). Sections 18.2.8 and 19.1.2 will derive them.

18.2.7 Convolution and correlation

In mechanical and electrical engineering, the concept of *convolution* emerges during the analysis of a linear system whose response to a Dirac impulse $\delta(t)$ is some characteristic *transfer function* $h(t)$. To explore the mechanical origin and engineering application of the transfer function would exceed the

[15]If working out this metadual with your own pencil according to the pattern of § 18.2.5, if you reach

$$\begin{aligned} G(v) &= \frac{d}{dv}F(v), \\ g(-v) &= ivf(-v), \end{aligned}$$

then you are probably on the right track. Note that, during the calculation of this particular metadual, it happens that $w \equiv v$ and $u \equiv v$, so this particular metadual is a little easier than some others to work out.

For a harder metadual, see § 18.2.7.

book's remit; but, inasmuch as a system is linear and its response to a Dirac impulse $\delta(t)$ is indeed $h(t)$, its response to an arbitrary input $f(t)$ cannot but be

$$g_1(t) \equiv \int_{-\infty}^{\infty} h(t-\tau) f(\tau)\, d\tau; \tag{18.31}$$

or, changing $t/2 + \tau \leftarrow \tau$ to improve the equation's symmetry,

$$g_1(t) \equiv \int_{-\infty}^{\infty} h\left(\frac{t}{2} - \tau\right) f\left(\frac{t}{2} + \tau\right) d\tau. \tag{18.32}$$

This integral defines[16] convolution of the two functions $f(t)$ and $h(t)$.

Changing $v \leftarrow t$ and $\psi \leftarrow \tau$ in (18.32) to comport with the notation found elsewhere in this section and then applying (18.4) yields that[17]

$$\begin{aligned}
&\mathscr{F}\left\{\int_{-\infty}^{\infty} h\left(\frac{v}{2}-\psi\right) f\left(\frac{v}{2}+\psi\right) d\psi\right\} \\
&\quad= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\theta} \int_{-\infty}^{\infty} h\left(\frac{\theta}{2}-\psi\right) f\left(\frac{\theta}{2}+\psi\right) d\psi\, d\theta \\
&\quad= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-iv\theta} h\left(\frac{\theta}{2}-\psi\right) f\left(\frac{\theta}{2}+\psi\right) d\theta\, d\psi.
\end{aligned}$$

Now changing $\phi \leftarrow \theta/2 + \psi$ within the inner integral,

$$\begin{aligned}
&\mathscr{F}\left\{\int_{-\infty}^{\infty} h\left(\frac{v}{2}-\psi\right) f\left(\frac{v}{2}+\psi\right) d\psi\right\} \\
&\quad= \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-iv(2\phi-2\psi)} h(\phi-2\psi) f(\phi)\, d\phi\, d\psi \\
&\quad= \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\phi} f(\phi) \int_{-\infty}^{\infty} e^{-iv(\phi-2\psi)} h(\phi-2\psi)\, d\psi\, d\phi.
\end{aligned}$$

Again changing $\mu \leftarrow \phi - 2\psi$ within the inner integral,

$$\begin{aligned}
&\mathscr{F}\left\{\int_{-\infty}^{\infty} h\left(\frac{v}{2}-\psi\right) f\left(\frac{v}{2}+\psi\right) d\psi\right\} \\
&\quad= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\phi} f(\phi) \int_{-\infty}^{\infty} e^{-iv\mu} h(\mu)\, d\mu\, d\phi \\
&\quad= \left[\sqrt{2\pi}\right] \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\mu} h(\mu)\, d\mu\right] \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\phi} f(\phi)\, d\phi\right] \\
&\quad= \left(\sqrt{2\pi}\right) H(v) F(v).
\end{aligned}$$

[16] [83, § 2.2]

[17] See § 17.4.1.

That is,

$$\int_{-\infty}^{\infty} h\left(\frac{v}{2}-\psi\right) f\left(\frac{v}{2}+\psi\right) d\psi \xrightarrow{\mathscr{F}} \left(\sqrt{2\pi}\right) H(v)F(v). \tag{18.33}$$

Symbolizing (18.33) in the manner of § 18.2.5 as

$$\gamma(v) \xrightarrow{\mathscr{F}} \Gamma(v),$$

where

$$\begin{aligned}
\gamma(v) &\equiv \int_{-\infty}^{\infty} \chi\left(\frac{v}{2}-\psi\right) \phi\left(\frac{v}{2}+\psi\right) d\psi, \\
\Gamma(v) &= \left(\sqrt{2\pi}\right) \mathrm{X}(v)\Phi(v), \\
\gamma(-v) &= \int_{-\infty}^{\infty} \chi\left(-\frac{v}{2}-\psi\right) \phi\left(-\frac{v}{2}+\psi\right) d\psi, \\
\Gamma(-v) &= \left(\sqrt{2\pi}\right) \mathrm{X}(-v)\Phi(-v), \\
w_\chi &\equiv \frac{v}{2}-\psi, \\
w_\phi &\equiv \frac{v}{2}+\psi, \\
u &\equiv v,
\end{aligned}$$

or, after changing symbols downward a row with respect to each of (18.27) and (18.28),

$$G(v) \xrightarrow{\mathscr{F}} g(-v),$$

where

$$\begin{aligned}
G(v) &= \int_{-\infty}^{\infty} H\left(\frac{v}{2}-\psi\right) F\left(\frac{v}{2}+\psi\right) d\psi, \\
g(-v) &= \left(\sqrt{2\pi}\right) h(-v)f(-v), \\
G(-v) &= \int_{-\infty}^{\infty} H\left(-\frac{v}{2}-\psi\right) F\left(-\frac{v}{2}+\psi\right) d\psi, \\
g(v) &= \left(\sqrt{2\pi}\right) h(v)f(v),
\end{aligned}$$

one finds the metadual $g(v) \xrightarrow{\mathscr{F}} G(v)$ of (18.33) to be

$$h(v)f(v) \xrightarrow{\mathscr{F}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H\left(\frac{v}{2}-\psi\right) F\left(\frac{v}{2}+\psi\right) d\psi. \tag{18.34}$$

Whether by (18.33) or by (18.34), convolution in the one domain evidently transforms to multiplication in the other.

Closely related to the convolutional integral (18.32) is the integral

$$g_2(t) \equiv \int_{-\infty}^{\infty} h\left(\tau - \frac{t}{2}\right) f\left(\tau + \frac{t}{2}\right) d\tau, \tag{18.35}$$

whose transform and dual transform are computed as above, with one extra step using (18.21), to be

$$\begin{aligned}\int_{-\infty}^{\infty} h\left(\psi - \frac{v}{2}\right) f\left(\psi + \frac{v}{2}\right) d\psi &\stackrel{\mathscr{F}}{\rightarrow} \left(\sqrt{2\pi}\right) H(-v)F(v),\\ h(-v)f(v) &\stackrel{\mathscr{F}}{\rightarrow} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H\left(\psi - \frac{v}{2}\right) F\left(\psi + \frac{v}{2}\right) d\psi.\end{aligned} \tag{18.36}$$

Furthermore, according to (18.17), $h^*(t) \stackrel{\mathscr{F}}{\rightarrow} H^*(-\omega^*)$, so

$$\begin{aligned}\int_{-\infty}^{\infty} h^*\left(\psi - \frac{v}{2}\right) f\left(\psi + \frac{v}{2}\right) d\psi &\stackrel{\mathscr{F}}{\rightarrow} \left(\sqrt{2\pi}\right) H^*(v^*)F(v),\\ h^*(-v)f(v) &\stackrel{\mathscr{F}}{\rightarrow} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H^*\left(\frac{v^*}{2} - \psi\right) F\left(\frac{v}{2} + \psi\right) d\psi;\end{aligned} \tag{18.37}$$

and indeed one can do the same to the transforms (18.33) and (18.34) of the convolutional integral, obtaining

$$\begin{aligned}\int_{-\infty}^{\infty} h^*\left(\frac{v}{2} - \psi\right) f\left(\frac{v}{2} + \psi\right) d\psi &\stackrel{\mathscr{F}}{\rightarrow} \left(\sqrt{2\pi}\right) H^*(-v^*)F(v),\\ h^*(v)f(v) &\stackrel{\mathscr{F}}{\rightarrow} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H^*\left(\psi - \frac{v^*}{2}\right) F\left(\psi + \frac{v}{2}\right) d\psi.\end{aligned} \tag{18.38}$$

(The v^* of eqns. 18.37 and 18.38 seems to imply that the argument v might be complex. Though this writer has encountered applications with complex h and f, and though complex H and F are the norm, the writer has never yet met an application with complex v. How to interpret the case of complex v, or whether such a case is even valid in Fourier work, are questions left open to the reader's consideration. It is perhaps interesting that H. F. Davis, author of a book on Fourier mathematics, does not in his book seem to consider the transform of a function whose argument is complex at all.[18] Still, it appears that one can consider complex $v \neq v^*$ at least in a formal sense, as in § 18.2.1; yet in applications at any rate, normally and maybe always, $v = v^*$ will be real.)

Unlike the operation the integral (18.32) expresses, known as convolution, the operation the integral (18.35) expresses has no special name as far as the writer is aware. However, the operation its variant

$$g_3(t) \equiv \int_{-\infty}^{\infty} h^* \left(\tau - \frac{t}{2}\right) f\left(\tau + \frac{t}{2}\right) d\tau \tag{18.39}$$

expresses does have a name. It is called *correlation,*[19] being a measure of the degree to which one function tracks another with an offset in the independent variable. Reviewing this subsection, we see that in (18.37) we have already determined the transform of the correlational integral (18.39). Moreover, assuming that $\Im(t) = 0$, we see in (18.38)'s second line that we have already determined the dual of this transform, as well. Convolution and correlation arise often enough in applications to enjoy their own, peculiar notations[20]

$$h(t) * f(t) \equiv \int_{-\infty}^{\infty} h\left(\frac{t}{2} - \tau\right) f\left(\frac{t}{2} + \tau\right) d\tau \tag{18.40}$$

for convolution and

$$R_{fh}(t) \equiv \int_{-\infty}^{\infty} h^* \left(\tau - \frac{t}{2}\right) f\left(\tau + \frac{t}{2}\right) d\tau \tag{18.41}$$

for correlation (in the latter of which one can read the symbol R_{fh} as "the correlation of f against h").

Nothing prevents one from correlating a function against itself, incidentally. The *autocorrelation*

$$R_{ff}(t) = \int_{-\infty}^{\infty} f^* \left(\tau - \frac{t}{2}\right) f\left(\tau + \frac{t}{2}\right) d\tau \tag{18.42}$$

proves useful at times.[21] For convolution, commutative and associative properties that

$$\begin{aligned} f(t) * h(t) &= h(t) * f(t), \\ f(t) * [g(t) * h(t)] &= [f(t) * g(t)] * h(t), \end{aligned} \tag{18.43}$$

[18] [43, § 6.7]

[19] Also called *cross-correlation,* as in [177]. The more authoritative [125, eqn. 10-47], which also prefixes the term with "cross-," introduces the notation here used, though normalized to a different domain of scale.

[20] [89, § 19.4]

[21] [82, § 1.6A]

may be demonstrated, the former by changing $-\tau \leftarrow \tau$ in (18.40) and the latter by Fourier transformation as $f(v) * [g(v) * h(v)] \xrightarrow{\mathscr{F}} (\sqrt{2\pi})F(v) \times [(\sqrt{2\pi})G(v)H(v)] = (\sqrt{2\pi})[(\sqrt{2\pi})F(v)G(v)]H(v) \xrightarrow{\mathscr{F}^{-1}} [f(v) * g(v)] * h(v)$.

In most cases of practical interest in applications, v is probably real even when $H(v)$ and $F(v)$ are not, so one can use (18.42) to write a transform pair like the first line of (18.37) in the style of

$$R_{ff}(t) \xrightarrow{\mathscr{F}} \left(\sqrt{2\pi}\right) |F(\omega)|^2, \quad \Im(t) = 0. \tag{18.44}$$

Electrical engineers call the quantity $|F(\omega)|^2$ on (18.44)'s right the *energy spectral density* of $f(t)$.[22] Equation (18.44) is significant among other reasons because, in electronic signaling—especially where the inevitable imposition of random environmental *noise* has degraded the signal—it may happen that an adequate estimate of $R_{ff}(t)$ is immediately available while sufficient information regarding $F(\omega)$ is unavailable. When this is the case, (18.44) affords an elegant, indirect way to calculate an energy spectral density even if more direct methods cannot be invoked.

Tables 18.2 and 18.3 summarize.

See also § 19.1.

18.2.8 Parseval's theorem

Provided that

$$\Im(v) = 0,$$

one finds that

$$\begin{aligned}
\int_{-\infty}^{\infty} h^*(v) f(v)\,dv &= \int_{-\infty}^{\infty} h^*(v) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iv\theta} F(\theta)\,d\theta \right] dv \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iv\theta} h^*(v)\,dv \right] F(\theta)\,d\theta \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\theta} h(v)\,dv \right]^* F(\theta)\,d\theta \\
&= \int_{-\infty}^{\infty} H^*(\theta) F(\theta)\,d\theta,
\end{aligned}$$

in which an interchange of integrations between two applications of (18.4) has been interposed. Changing $v \leftarrow \theta$ on the right,

$$\int_{-\infty}^{\infty} h^*(v) f(v)\,dv = \int_{-\infty}^{\infty} H^*(v) F(v)\,dv. \tag{18.45}$$

[22] [82, § 1.6B]

Table 18.2: Convolution and correlation, and their Fourier properties. (Note that, though the table provides for complex v, v is typically real.)

$$\begin{aligned}
\int_{-\infty}^{\infty} h\left(\frac{v}{2}-\psi\right) f\left(\frac{v}{2}+\psi\right) d\psi &\xrightarrow{\mathscr{F}} \left(\sqrt{2\pi}\right) H(v)F(v) \\
h(v)f(v) &\xrightarrow{\mathscr{F}} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H\left(\frac{v}{2}-\psi\right) \\
&\qquad\qquad \times F\left(\frac{v}{2}+\psi\right) d\psi \\
\int_{-\infty}^{\infty} h\left(\psi-\frac{v}{2}\right) f\left(\psi+\frac{v}{2}\right) d\psi &\xrightarrow{\mathscr{F}} \left(\sqrt{2\pi}\right) H(-v)F(v) \\
h(-v)f(v) &\xrightarrow{\mathscr{F}} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H\left(\psi-\frac{v}{2}\right) \\
&\qquad\qquad \times F\left(\psi+\frac{v}{2}\right) d\psi \\
\int_{-\infty}^{\infty} h^*\left(\frac{v}{2}-\psi\right) f\left(\frac{v}{2}+\psi\right) d\psi &\xrightarrow{\mathscr{F}} \left(\sqrt{2\pi}\right) H^*(-v^*)F(v) \\
h^*(v)f(v) &\xrightarrow{\mathscr{F}} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H^*\left(\psi-\frac{v^*}{2}\right) \\
&\qquad\qquad \times F\left(\psi+\frac{v}{2}\right) d\psi \\
\int_{-\infty}^{\infty} h^*\left(\psi-\frac{v}{2}\right) f\left(\psi+\frac{v}{2}\right) d\psi &\xrightarrow{\mathscr{F}} \left(\sqrt{2\pi}\right) H^*(v^*)F(v) \\
h^*(-v)f(v) &\xrightarrow{\mathscr{F}} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H^*\left(\frac{v^*}{2}-\psi\right) \\
&\qquad\qquad \times F\left(\frac{v}{2}+\psi\right) d\psi
\end{aligned}$$

Table 18.3: Convolution and correlation in their peculiar notation. (Note that the $*$ which appears in the table as $h[t] * f[t]$ differs in meaning from the * in $h^*[v]$.)

$$
\begin{aligned}
\Im(t) &= 0 \\
f(t) * h(t) = h(t) * f(t) &\equiv \int_{-\infty}^{\infty} h\left(\frac{t}{2} - \tau\right) f\left(\frac{t}{2} + \tau\right) d\tau \\
R_{fh}(t) &\equiv \int_{-\infty}^{\infty} h^*\left(\tau - \frac{t}{2}\right) f\left(\tau + \frac{t}{2}\right) d\tau \\
h(t) * f(t) &\stackrel{\mathscr{F}}{\to} \left(\sqrt{2\pi}\right) H(\omega)F(\omega) \\
h(t)f(t) &\stackrel{\mathscr{F}}{\to} \frac{1}{\sqrt{2\pi}}[H(\omega) * F(\omega)] \\
R_{fh}(t) &\stackrel{\mathscr{F}}{\to} \left(\sqrt{2\pi}\right) H^*(\omega)F(\omega) \\
h^*(t)f(t) &\stackrel{\mathscr{F}}{\to} \frac{1}{\sqrt{2\pi}} R_{FH}(\omega) \\
R_{ff}(t) &\stackrel{\mathscr{F}}{\to} \left(\sqrt{2\pi}\right) |F(\omega)|^2 \\
f(t) * [g(t) * h(t)] &= [f(t) * g(t)] * h(t)
\end{aligned}
$$

This is *Parseval's theorem.*[23] It is related to Parseval's principle of § 17.1 and Parseval's equality of § 17.5.1.

Especially interesting to Parseval's theorem is the case of $h(v) = f(v)$, in which

$$\int_{-\infty}^{\infty} |f(v)|^2 \, dv = \int_{-\infty}^{\infty} |F(v)|^2 \, dv. \tag{18.46}$$

When this is written as

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega,$$

and t, $|f(t)|^2$, ω and $|F(\omega)|^2$ respectively have physical dimensions of time, energy per unit time, angular frequency and energy per unit angular frequency, then the theorem conveys the important physical insight that energy transferred at various times can equally well be regarded as energy transferred *at various frequencies.* This works for space and spatial frequencies, too: see § 17.2. For real $f(v)$, one can write (18.46) as

$$\int_{-\infty}^{\infty} f^2(v)\, dv = \int_{-\infty}^{\infty} \Re^2[F(v)]\, dv + \int_{-\infty}^{\infty} \Im^2[F(v)]\, dv, \quad \Im[f(v)] = 0, \tag{18.47}$$

which expresses the principle of *quadrature,* conveying the additional physical insight that a single frequency can carry energy in not only one but each of two distinct, independent channels; namely, a *real-phased, in-phase* or I channel and an *imaginary-phased, quadrature-phase* or Q channel.[24] Practical digital electronic communications systems, wired or wireless, often do precisely this—effectively transmitting two, independent streams of information at once, without conflict, in the selfsame band.

18.2.9 Oddness and evenness

Odd functions have odd transforms. Even functions have even transforms. Symbolically,

- [ODD] if $f(-v) = -f(v)$ for all v, then $F(-v) = -F(v)$;
- [EVEN] if $f(-v) = f(v)$ for all v, then $F(-v) = F(v)$.

The odd case is proved by expressing $F(-v)$ per (18.4) as

$$F(-v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(-v)\theta} f(\theta)\, d\theta$$

[23] [37, § 2-2][82, § 1.6B]

[24] [37, § 5-1]

and then changing $-\theta \leftarrow \theta$ to get that

$$F(-v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(-v)(-\theta)} f(-\theta)\, d\theta = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\theta} f(-\theta)\, d\theta.$$

That $f(v)$ should be odd means by definition that $f(-\theta) = -f(\theta)$, so

$$F(-v) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iv\theta} f(\theta)\, d\theta = -F(v).$$

The even case is proved similarly. See § 8.12.

18.3 The Fourier transforms of selected functions

We have already computed the Fourier transforms of several functions in §§ 18.1.4 and 18.2.3. We have also already computed duals of these but would still like to put the duals into more pleasing forms. For example, the dual of

$$\Lambda(v) \xrightarrow{\mathscr{F}} \frac{\mathrm{Sa}^2(v/2)}{\sqrt{2\pi}}$$

according to (18.22) is

$$\frac{\mathrm{Sa}^2(v/2)}{\sqrt{2\pi}} \xrightarrow{\mathscr{F}} \Lambda(v).$$

The scaling property (18.29) of Table 18.1, using $\alpha = 2$, makes it

$$\mathrm{Sa}^2\, v \xrightarrow{\mathscr{F}} \frac{\sqrt{2\pi}}{2} \Lambda\left(\frac{v}{2}\right),$$

which is probably more convenient than (18.22) to use when one meets a $\mathrm{Sa}^2(\cdot)$ and wishes to transform it.

Tables 18.4 and 18.5 list the last transform pair and others similarly derived.[25] The tables list further transform pairs as well—some as gleaned from earlier in the chapter; others as computed in the last paragraph's way, as adapted by the properties of Table 18.1 (especially the properties of delay, shift and scaling), or as derived in the subsections that follow. The final pair of Table 18.5 is derived in § 18.4.

[25] In electronic signaling systems, including radio, the table's transform pair $\mathrm{Sa}(v) \xrightarrow{\mathscr{F}} \frac{\sqrt{2\pi}}{2}\Pi\left(\frac{v}{2}\right)$ implies significantly that, to spread energy evenly over an available "baseband" but to let no energy leak outside that band, one should transmit sine-argument-shaped pulses as in Fig. 17.10.

Table 18.4: Fourier transform pairs. (See also Table 18.5.)

$$1 \stackrel{\mathscr{F}}{\to} \left(\sqrt{2\pi}\right)\delta(v) \qquad \delta(v) \stackrel{\mathscr{F}}{\to} \frac{1}{\sqrt{2\pi}}$$

$$\Lambda(v) \stackrel{\mathscr{F}}{\to} \frac{\mathrm{Sa}^2(v/2)}{\sqrt{2\pi}} \qquad u(v) \stackrel{\mathscr{F}}{\to} \frac{1}{\left(\sqrt{2\pi}\right) iv} + \frac{\sqrt{2\pi}}{2}\delta(v)$$

$$\Lambda_r(v) \stackrel{\mathscr{F}}{\to} \frac{\mathrm{Sa}(rv/2)\,\mathrm{Sa}(v/2)}{\sqrt{2\pi}} \qquad \Pi(v) \stackrel{\mathscr{F}}{\to} \frac{\mathrm{Sa}(v/2)}{\sqrt{2\pi}}$$

$$\Psi(v) \stackrel{\mathscr{F}}{\to} \frac{\mathrm{Sa}\,v}{\sqrt{2\pi}\,[1-(v/\pi)^2]} \qquad \mathscr{F}\{\Psi(v)\}_{v=\pm\pi} = \frac{1}{2\sqrt{2\pi}}$$

$$\Psi_r(v) \stackrel{\mathscr{F}}{\to} \frac{\cos(rv/2)\,\mathrm{Sa}(v/2)}{\sqrt{2\pi}\,[1-(rv/\pi)^2]} \qquad \mathscr{F}\{\Psi_r(v)\}_{rv=\pm\pi} = \frac{\left(\sqrt{2\pi}\right)\mathrm{Sa}(v/2)}{8} = \frac{\left(\sqrt{2\pi}\right)\mathrm{Sa}(2\pi/4r)}{8}$$

$$\mathrm{Sa}^2(v) \stackrel{\mathscr{F}}{\to} \frac{\sqrt{2\pi}}{2}\Lambda\left(\frac{v}{2}\right) \qquad \mathrm{Sa}(v) \stackrel{\mathscr{F}}{\to} \frac{\sqrt{2\pi}}{2}\Pi\left(\frac{v}{2}\right)$$

$$u(v)e^{-av} \stackrel{\mathscr{F}}{\to} \frac{1}{\left(\sqrt{2\pi}\right)(a+iv)}, \qquad \Re(a) > 0$$

$$u(v)e^{-av}v^n \stackrel{\mathscr{F}}{\to} \frac{n!}{\left(\sqrt{2\pi}\right)(a+iv)^{n+1}}, \quad \Re(a) > 0,\ n \in \mathbb{Z},\ n \ge 0$$

$$e^{iav} \stackrel{\mathscr{F}}{\to} \left(\sqrt{2\pi}\right)\delta(v-a), \qquad \Im(a) = 0$$

$$\sin av \stackrel{\mathscr{F}}{\to} \frac{\sqrt{2\pi}}{i2}\left[\delta(v-a) - \delta(v+a)\right], \quad \Im(a) = 0$$

$$\cos av \stackrel{\mathscr{F}}{\to} \frac{\sqrt{2\pi}}{2}\left[\delta(v-a) + \delta(v+a)\right], \quad \Im(a) = 0$$

$$\sum_{j=-\infty}^{\infty} \delta(v - jT_1) \stackrel{\mathscr{F}}{\to} \frac{\sqrt{2\pi}}{T_1}\sum_{j=-\infty}^{\infty} \delta\left(v - j\frac{2\pi}{T_1}\right) = \frac{\Delta\omega}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} \delta(v - j\,\Delta\omega)$$

$$\sqrt{T_1}\sum_{j=-\infty}^{\infty} \delta(v - jT_1) \stackrel{\mathscr{F}}{\to} \sqrt{\Delta\omega}\sum_{j=-\infty}^{\infty} \delta(v - j\,\Delta\omega), \quad \Delta\omega\, T_1 = 2\pi$$

Table 18.5: Fourier autotransform pairs.

$$\sum_{j=-\infty}^{\infty} \delta\left(v - j\sqrt{2\pi}\right) \stackrel{\mathscr{F}}{\rightarrow} \sum_{j=-\infty}^{\infty} \delta\left(v - j\sqrt{2\pi}\right)$$

$$\Omega(v) \stackrel{\mathscr{F}}{\rightarrow} \Omega(v)$$

18.3.1 Exponential decay and Heaviside's unit step

Application of the Fourier transform's definition (18.4) to $u(v)e^{-av}$, where $u(v)$ is Heaviside's unit step (7.21), yields that

$$\mathscr{F}\left\{u(v)e^{-av}\right\} = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-(a+iv)\theta}\, d\theta = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(a+iv)\theta}}{-(a+iv)}\right]_{\theta=0}^{\infty},$$

revealing the transform pair

$$u(v)e^{-av} \stackrel{\mathscr{F}}{\rightarrow} \frac{1}{\left(\sqrt{2\pi}\right)(a+iv)}, \quad \Re(a) > 0. \tag{18.48}$$

Interesting is the limit $a \to 0^+$ in (18.48),

$$u(v) \stackrel{\mathscr{F}}{\rightarrow} \frac{1}{\left(\sqrt{2\pi}\right)(iv)} + C\delta(v),$$

where the necessary term $C\delta(v)$, with scale C to be determined, merely admits that we do not yet know how to evaluate (18.48) when both a and v vanish at once. What we do know from § 18.2.9 is that odd functions have odd transforms and that (as one can see in Fig. 7.10) one can convert $u(v)$ to an odd function by the simple expedient of subtracting $1/2$ from it. Since $1/2 \stackrel{\mathscr{F}}{\rightarrow} (\sqrt{2\pi}/2)\delta(v)$ according to (18.24), we have then that

$$u(v) - \frac{1}{2} \stackrel{\mathscr{F}}{\rightarrow} \frac{1}{\left(\sqrt{2\pi}\right)(iv)} + \left(C - \frac{\sqrt{2\pi}}{2}\right)\delta(v),$$

which to make its right side odd demands that $C = \sqrt{2\pi}/2$. The transform pair

$$u(v) \stackrel{\mathscr{F}}{\rightarrow} \frac{1}{\left(\sqrt{2\pi}\right) iv} + \frac{\sqrt{2\pi}}{2}\delta(v) \tag{18.49}$$

results.

Invoking (18.4),

$$u(v)e^{-av}v^n \overset{\mathscr{F}}{\to} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(a+iv)\theta}\theta^n \, d\theta.$$

Evaluating the antiderivative via Table 9.1 with $\alpha \leftarrow -(a+iv)$,

$$u(v)e^{-av}v^n \overset{\mathscr{F}}{\to} -\frac{e^{-(a+iv)\theta}}{\sqrt{2\pi}} \sum_{k=0}^{n} \left.\frac{(n!/k!)\theta^k}{(a+iv)^{n-k+1}}\right|_{\theta=0}^{\infty}.$$

Since all but the $k = 0$ term vanish, the last equation implies the transform pair[26]

$$u(v)e^{-av}v^n \overset{\mathscr{F}}{\to} \frac{n!}{\sqrt{2\pi}(a+iv)^{n+1}}, \quad \Re(a) > 0,\ n \in \mathbb{Z},\ n \geq 0. \tag{18.50}$$

18.3.2 Sinusoids

The Fourier transforms of $\sin av$ and $\cos av$ are interesting and important. One can compute them from the pairs

$$\begin{aligned} e^{iav} &\overset{\mathscr{F}}{\to} \left(\sqrt{2\pi}\right)\delta(v-a), \\ e^{-iav} &\overset{\mathscr{F}}{\to} \left(\sqrt{2\pi}\right)\delta(v+a), \end{aligned} \tag{18.51}$$

which result by applying to (18.24) Table 18.1's property that $e^{iav}f(v) \overset{\mathscr{F}}{\to} F(v-a)$. Composing by Table 5.1 the trigonometrics from their complex parts, we have that

$$\begin{aligned} \sin av &\overset{\mathscr{F}}{\to} \frac{\sqrt{2\pi}}{i2}\left[\delta(v-a) - \delta(v+a)\right], \\ \cos av &\overset{\mathscr{F}}{\to} \frac{\sqrt{2\pi}}{2}\left[\delta(v-a) + \delta(v+a)\right]. \end{aligned} \tag{18.52}$$

18.3.3 The Dirac delta pulse train

Curiously, the Fourier transform of the Dirac delta pulse train of Fig. 17.8 turns out to be another Dirac delta pulse train. The reason is that the Dirac delta pulse train's Fourier series according to (17.28) and (17.21) is

$$\sum_{j=-\infty}^{\infty} \delta(v - jT_1) = \sum_{j=-\infty}^{\infty} \frac{e^{ij(2\pi/T_1)v}}{T_1},$$

[26] [129, Table 5.2]

the transform of which according to (18.51) is

$$\sum_{j=-\infty}^{\infty} \delta(v - jT_1) \xrightarrow{\mathscr{F}} \frac{\sqrt{2\pi}}{T_1} \sum_{j=-\infty}^{\infty} \delta\left(v - j\frac{2\pi}{T_1}\right). \tag{18.53}$$

Apparently, the farther the pulses of the original train, the nearer the pulses of the transformed train, and vice versa; yet, even when transformed, the train remains a train of Dirac deltas. Letting $T_1 = \sqrt{2\pi}$ in (18.53) yields the pair

$$\sum_{j=-\infty}^{\infty} \delta\left(v - j\sqrt{2\pi}\right) \xrightarrow{\mathscr{F}} \sum_{j=-\infty}^{\infty} \delta\left(v - j\sqrt{2\pi}\right), \tag{18.54}$$

discovering a pulse train whose Fourier transform is itself.

This completes the derivations of the Fourier transform pairs of Tables 18.4 and 18.5—except one pair. The one pair will be the subject of § 18.4, next.

18.4 The Gaussian pulse

While studying the derivative in chapters 4 and 5, we asked among other questions whether any function could be its own derivative. We found that a sinusoid could be its own derivative after a fashion—differentiation shifted its curve leftward without altering its shape or scale—but that the only nontrivial function to be exactly its own derivative was the natural exponential $f(z) = Ae^z$. We later found the same natural exponential to fill several significant mathematical roles—largely, whether directly or indirectly, because it was indeed its own derivative.

As we study the Fourier transform, a similar question arises: can any function be its own *transform?* We have already found in § 18.3.3 that the Dirac delta pulse train can be its own transform; but this train unlike the natural exponential is nonanalytic, perhaps not the sort of function one had in mind. One should like an analytic function, and preferably not a train but a single pulse.

In chapter 20, during the study of the mathematics of probability, we shall encounter a most curious function, the *Gaussian pulse,* also known as the *bell curve* among other names. We will defer discussion of the Gaussian pulse's provenance to the coming chapters but, for now, we can just copy here the pulse's definition from (20.17) as

$$\Omega(t) \equiv \frac{\exp\left(-t^2/2\right)}{\sqrt{2\pi}}, \tag{18.55}$$

plotted on pages 688 below and 589 above, respectively in Figs. 20.1 and 17.6. The Fourier transform of the Gaussian pulse is even trickier to compute than were the transforms of § 18.3, but known techniques to compute it include the following.[27] From the the Fourier transform's definition (18.4),

$$\mathscr{F}\{\Omega(v)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{\theta^2}{2} - iv\theta\right) d\theta.$$

Completing the square (§ 2.2),[28]

$$\begin{aligned}\mathscr{F}\{\Omega(v)\} &= \frac{\exp\left(-v^2/2\right)}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{\theta^2}{2} - iv\theta + \frac{v^2}{2}\right) d\theta \\ &= \frac{\exp\left(-v^2/2\right)}{2\pi}\int_{-\infty}^{\infty} \exp\left[-\frac{(\theta+iv)^2}{2}\right] d\theta.\end{aligned}$$

Changing $\xi \leftarrow \theta + iv$,

$$\mathscr{F}\{\Omega(v)\} = \frac{\exp\left(-v^2/2\right)}{2\pi}\int_{-\infty+iv}^{\infty+iv} \exp\left(-\frac{\xi^2}{2}\right) d\xi.$$

Had we not studied complex contour integration in § 9.6 we should find such an integral hard to integrate in closed form. However, since happily we have studied it, observing that the integrand $\exp(\xi^2)$ is an entire function (§ 8.6) of ξ—that is, that it is everywhere analytic—we recognize that one can trace the path of integration from $-\infty + iv$ to $\infty + iv$ along any contour one likes. Let us trace it along the real Argand axis from $-\infty$ to ∞, leaving only the two, short complex segments at the ends which (as is easy enough to see, and the formal proof is left as an exercise to the interested reader[29]) lie so far away that—for this integrand—they integrate to nothing. So tracing leaves us with

$$\mathscr{F}\{\Omega(v)\} = \frac{\exp\left(-v^2/2\right)}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{\xi^2}{2}\right) d\xi. \tag{18.56}$$

How to proceed from here is not immediately obvious. None of the techniques of chapter 9 seems especially suitable to evaluate

$$I \equiv \int_{-\infty}^{\infty} \exp\left(-\frac{\xi^2}{2}\right) d\xi,$$

[27] An alternate technique is outlined in [83, Prob. 5.43].

[28] [42]

[29] The short complex segments at the ends might integrate to something were the real part of ξ^2 negative, but the real part happens to be positive—indeed, most extremely positive—over the domains of the segments in question.

though if a search for a suitable contour of integration failed one might fall back on the Taylor-series technique of § 9.12. Fortunately, mathematicians have been searching hundreds of years for clever techniques to evaluate just such integrals and, when occasionally they should discover such a technique and reveal it to us, why, we record it in books like this, not to forget.

Here is the technique.[30] The equations

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx, \\ I &= \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy, \end{aligned}$$

express the same integral I in two different ways, the only difference being in the choice of letter for the dummy variable. What if we multiply the two? Then

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{x^2+y^2}{2}\right) dx\, dy. \end{aligned}$$

One, geometrical way to interpret this I^2 is as a double integration over a plane in which (x, y) are rectangular coordinates. If we interpret thus, nothing then prevents us from double-integrating by the cylindrical coordinates $(\rho; \phi)$, instead, as

$$\begin{aligned} I^2 &= \int_{-\pi}^{\pi}\int_{0}^{\infty} \exp\left(-\frac{\rho^2}{2}\right) \rho\, d\rho\, d\phi \\ &= 2\pi\left[\int_{0}^{\infty} \exp\left(-\frac{\rho^2}{2}\right) \rho\, d\rho\right]. \end{aligned}$$

At a casual glance, the last integral in square brackets seems to differ little from the integral with which we started, but see: it is not only that the lower limit of integration and the letter of the dummy variable have changed, but that an extra factor of the dummy variable has appeared—that the integrand ends not with $d\rho$ but with $\rho\, d\rho$. Once we have realized this, the integral's solution by antiderivative (§ 9.1) becomes suddenly easy to guess:

$$I^2 = 2\pi\left[-\exp\left(-\frac{\rho^2}{2}\right)\right]_0^{\infty} = 2\pi.$$

[30] [57, § I:40-4]

So evidently,

$$I = \sqrt{2\pi},$$

which means that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{\xi^2}{2}\right) d\xi = \sqrt{2\pi} \tag{18.57}$$

as was to be calculated.

Finally substituting (18.57) into (18.56), we have that

$$\mathscr{F}\{\Omega(v)\} = \frac{\exp\left(-v^2/2\right)}{\sqrt{2\pi}},$$

which in view of (18.55) reveals the remarkable transform pair

$$\Omega(v) \stackrel{\mathscr{F}}{\rightarrow} \Omega(v). \tag{18.58}$$

The Gaussian pulse transforms to itself. Old Fourier, who can twist and knot other curves with ease, seems powerless to bend Gauss' mighty curve.

It is worth observing incidentally in light of (18.55) and (18.57) that

$$\int_{-\infty}^{\infty} \Omega(t)\, dt = 1, \tag{18.59}$$

the same as for $\Pi(t)$, $\Lambda(t)$, $\Psi(t)$ and indeed $\delta(t)$. Section 17.3 and its (17.20) have recommended the shape of the Gaussian pulse, in the tall, narrow limit, to implement the Dirac delta $\delta(t)$. This section lends more force to the recommendation, for not only is the Gaussian pulse analytic (unlike the Dirac delta) but it also behaves uncommonly well under Fourier transformation (like the Dirac delta), thus rendering the Dirac delta susceptible to an analytic limiting process which transforms amenably. Too, the Gaussian pulse is about as tightly localized as a nontrivial, uncontrived analytic function can be.[31] The passion of one of the author's mentors in extolling the Gaussian pulse as "absolutely a beautiful function" seems well supported by the practical mathematical virtues exhibited by the function itself.

The Gaussian pulse resembles the natural exponential in its general versatility. Indeed, though the book has required several chapters through this chapter 18 to develop the fairly deep mathematics underlying the Gaussian pulse and supporting its basic application, now that we have the Gaussian pulse in hand we shall find that it ably fills all sorts of roles—not least the principal role of chapter 20 to come.

[31] Consider that $\Omega(t) \approx \mathrm{0x0.6621}, \mathrm{0x0.3DF2}, \mathrm{0x0.0DD2}, \mathrm{0x0.0122}, \mathrm{0x0.0009}, \mathrm{0x0.0000}$ at $t = 0, \pm1, \pm2, \pm3, \pm4, \pm5$; and that $\Omega(\pm8) < 2^{-\mathrm{0x2F}}$. Away from its middle region $|t| \lesssim 1$, the Gaussian pulse evidently vanishes rather convincingly.

18.5 The Fourier transform of the integration operation

Table 18.1, page 630, includes a heretofore unproved Fourier property,

$$\int_{-\infty}^{v} f(\tau)\,d\tau \xrightarrow{\mathscr{F}} \frac{F(v)}{iv} + \pi F(0)\delta(v). \tag{18.60}$$

This property has remained unproved because, when we compiled the table in § 18.2, we lacked the needed theory. We have the theory now and can proceed with the proof.[32]

The proof begins with the observation that

$$\int_{-\infty}^{v} f(\tau)\,d\tau = u(v) * f(v) = \int_{-\infty}^{\infty} u\left(\frac{v}{2}-\tau\right) f\left(\frac{v}{2}+\tau\right) d\tau,$$

where the $u(t)*f(t)$ exercises the convolution operation of § 18.2.7. The convolution's correctness is probably easier to see if the convolution is expressed according to (18.31) rather than to (18.32), as

$$\int_{-\infty}^{v} f(\tau)\,d\tau = u(v) * f(v) = \int_{-\infty}^{\infty} u(v-\tau) f(\tau)\,d\tau;$$

but the two forms are equivalent and, since we are free to work with either and the earlier is the form that appears in Table 18.2, we will prefer the earlier form.

Table 18.2 records the transform pair

$$\begin{aligned}&\int_{-\infty}^{\infty} u\left(\frac{v}{2}-\tau\right) f\left(\frac{v}{2}+\tau\right) d\tau \\ &\quad\xrightarrow{\mathscr{F}} \quad \left(\sqrt{2\pi}\right) \mathscr{F}\{u(v)\}F(v) = \left[\frac{1}{iv} + \pi\delta(v)\right] F(v)\end{aligned}$$

in which $\mathscr{F}\{u(v)\}$ is evaluated according to Table 18.4. Substituting the observation with which we began,

$$\int_{-\infty}^{v} f(\tau)\,d\tau = u(v) * f(v) \xrightarrow{\mathscr{F}} \left[\frac{1}{iv} + \pi\delta(v)\right] F(v).$$

Sifting, $\delta(v)F(v) = \delta(v)F(0)$, so the last pair is in fact (18.60) which was to be proved.

---

[32] [83, Prob. 5.33]

A fact that has incidentally emerged during the proof,

$$\int_{-\infty}^{v} f(\tau)\,d\tau = u(v) * f(v), \tag{18.61}$$

is interesting enough to merit here an equation number of its own.

Chapter 19

Fourier applications and the Laplace transform

The last chapter has unveiled the Fourier transform. This chapter pursues some Fourier applications. The chapter also introduces the Laplace transform, a Fourier variant used to solve ordinary linear differential equations constrained by boundary conditions.

The chapter begins with a review of convolution.

19.1 Convolution

Section 18.2.7 and its (18.40) have mentioned the *convolution* of two functions,

$$h(t) * f(t) \equiv \int_{-\infty}^{\infty} h\left(\frac{t}{2} - \tau\right) f\left(\frac{t}{2} + \tau\right) d\tau, \tag{19.1}$$

and have expressed this convolution also in unbalanced style as

$$h(t) * f(t) \equiv \int_{-\infty}^{\infty} h(t - \tau) f(\tau)\, d\tau, \tag{19.2}$$

the two styles—whether (19.1) or (19.2)—being equivalent. The two styles express the same operation and yield the same result.

Convolution is commutative and associative as (18.43) has observed:

$$\begin{aligned} f(t) * h(t) &= h(t) * f(t); \\ f(t) * [g(t) * h(t)] &= [f(t) * g(t)] * h(t). \end{aligned}$$

Despite exploring a few properties like these, though, and despite giving a formal definition, we have developed little insight into the convolution operation. What *is* convolution, really?

19.1.1 What convolution is

Several answers are possible. In this book, our answer will chiefly be this:

> CONVOLUTION IS THE SUPERPOSITION OF REPETITION WITH VARIOUS SCALE AND DELAY.

Well, that's fine. The answer is dense with multisyllabic words, at any rate. What does it mean?

To begin, a *superposition* is just a sum, especially a sum of functions. The word comes from the Latin[1] *super* + *positio*, "above placement," and connotes a laying of one atop another, as bricks in a wall. Figure 17.2 of an earlier chapter has plotted a typical case.

Regarding "repetition with various scale and delay," an example:[2]

$$\begin{aligned} f(t) &= (1)\,\delta\left[t - \left(-\frac{\mathrm{0x18}}{5}\right)\right] + (1)\,\delta\left[t - \left(-\frac{5}{2}\right)\right] + \left(\frac{3}{2}\right)\delta\left[t - 0\right] \\ &\qquad + \left(\frac{2}{3}\right)\delta\left[t - \frac{1}{2}\right] + \left(-\frac{8}{3}\right)\delta\left[t - \frac{\mathrm{0xB}}{4}\right]; \\ h(t) &= \Lambda(t). \end{aligned}$$

Here, the 1, 1, $\frac{3}{2}$, $\frac{2}{3}$ and $-\frac{8}{3}$ give scale, while the $-\frac{\mathrm{0x18}}{5}$, $-\frac{5}{2}$, 0, $\frac{1}{2}$ and $\frac{\mathrm{0xB}}{4}$ give delay. Convolving according to (19.2) and sifting by (7.25),

$$\begin{aligned} h(t) * f(t) &= (1)\,\Lambda\left[t - \left(-\frac{\mathrm{0x18}}{5}\right)\right] + (1)\,\Lambda\left[t - \left(-\frac{5}{2}\right)\right] + \left(\frac{3}{2}\right)\Lambda\left[t - 0\right] \\ &\qquad + \left(\frac{2}{3}\right)\Lambda\left[t - \frac{1}{2}\right] + \left(-\frac{8}{3}\right)\Lambda\left[t - \frac{\mathrm{0xB}}{4}\right]. \end{aligned}$$

Figure 19.1 plots.

But what if $f(t)$ is no mere sum of Dirac deltas[3] but a more general function? In that case, one can approximate that

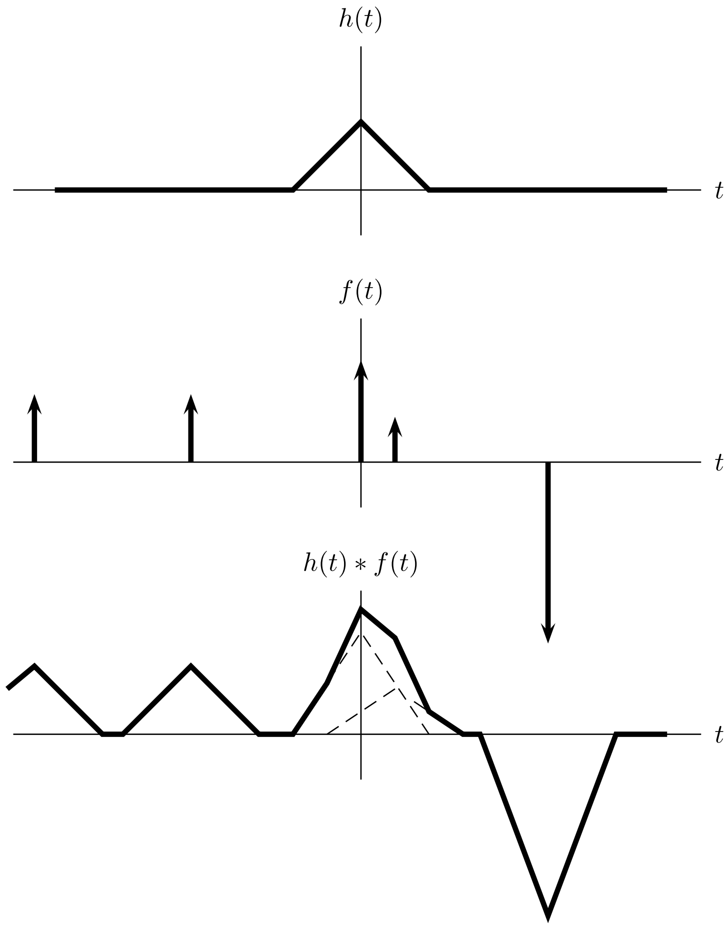
$$f(t) = \lim_{\Delta t \to 0} \Delta t \sum_{k=-\infty}^{\infty} f(k\,\Delta t)\delta(t - k\,\Delta t); \tag{19.3}$$

[1] [134]

[2] Patterned after [128].

[3] Mere! How use accustoms one to the ephemeral!

Figure 19.1: Convolution.



or, if unsure that one can trust (19.3) as written, that

$$f(t) = \lim_{\Delta t \to 0} \sum_{k=-\infty}^{\infty} f(k\,\Delta t)\Pi\left(\frac{t - k\,\Delta t}{\Delta t}\right),$$

which, in view of § 7.7, says the same thing. Processing (19.3) by Fig. 19.1's method,

$$h(t) * f(t) = \lim_{\Delta t \to 0} \Delta t \sum_{k=-\infty}^{\infty} f(k\,\Delta t)h(t - k\,\Delta t).$$

19.1.2 Series coefficients

Given a time-limited pulse

$$f(t) = 0 \text{ for all } |t - t_o| \geq \frac{T_1}{2},$$

one can write that

$$\int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt = \int_{t_o - T_1/2}^{t_o + T_1/2} e^{-i\omega t} f(t)\,dt,$$

after which comparison of the Fourier transform (18.5) against the coefficients (17.22) of the Fourier *series* reveals that

$$a_j = \frac{\Delta\omega}{\sqrt{2\pi}} F(j\,\Delta\omega), \tag{19.4}$$

where $\Delta\omega\,T_1 = 2\pi$ as in (17.3). If a time-limited pulse repeats at an interval T_1 to form a wave and the interval is long enough that the wave's pulses do not overlap, then (19.4) extracts the coefficients of the waveform's Fourier series from the transform of the pulse.

If pulses overlap then one must do extra work to reach the same result, representing the repeating waveform via (19.2) and (7.25) as

$$\hat{f}(t) \equiv \sum_{k=-\infty}^{\infty} f(t - kT_1) = \left[\sum_{k=-\infty}^{\infty} \delta(t - kT_1)\right] * f(t). \tag{19.5}$$

The summation in (19.5) might diverge, in which case the repeating waveform would not exist, but if the summation does not diverge then (19.4) still obtains despite the overlap. To prove it, beginning from (17.22) by successive steps,[4]

$$
\begin{aligned}
a_j &= \frac{1}{T_1}\int_{t_o-T_1/2}^{t_o+T_1/2} e^{-ij\,\Delta\omega\,\tau}\hat{f}(\tau)\,d\tau \\
&= \frac{1}{T_1}\int_{-\infty}^{\infty} \Pi\left(\frac{\tau-t_o}{T_1}\right) e^{-ij\,\Delta\omega\,\tau}\hat{f}(\tau)\,d\tau \\
&= \frac{\sqrt{2\pi}}{T_1}\left[\mathscr{F}\left\{\Pi\left(\frac{t-t_o}{T_1}\right)\hat{f}(t)\right\}\right]_{\omega=j\,\Delta\omega} \\
&= \frac{1}{T_1}\left[\mathscr{F}\left\{\Pi\left(\frac{t-t_o}{T_1}\right)\right\} * \mathscr{F}\left\{\hat{f}(t)\right\}\right]_{\omega=j\,\Delta\omega} \\
&= \frac{1}{T_1}\left[\left(e^{-i\omega t_o}\mathscr{F}\left\{\Pi\left(\frac{t}{T_1}\right)\right\}\right) * \mathscr{F}\left\{\hat{f}(t)\right\}\right]_{\omega=j\,\Delta\omega},
\end{aligned}
$$

in which we have used the square pulse of § 17.3 and some properties of Tables 18.1 and 18.3. Finding in Table 18.4 the transform pair $\Pi(v) \stackrel{\mathscr{F}}{\rightarrow} \mathrm{Sa}(v/2)/\sqrt{2\pi}$, applying (18.29) to this pair with $\alpha = 1/T_1$, observing that $\Delta\omega\, T_1 = 2\pi$, and changing $\mathscr{F}_{vv} \leftarrow \mathscr{F}_{\omega t}$ yields the modified pair

$$
\Pi\left(\frac{t}{T_1}\right) \stackrel{\mathscr{F}}{\rightarrow} \frac{\left(\sqrt{2\pi}\right)\mathrm{Sa}(2\pi\omega/2\,\Delta\omega)}{\Delta\omega}, \tag{19.6}
$$

which, when used against the last expression for a_j, gives that

$$
\begin{aligned}
a_j &= \frac{1}{T_1}\left[\frac{e^{-i\omega t_o}\left(\sqrt{2\pi}\right)\mathrm{Sa}(2\pi\omega/2\,\Delta\omega)}{\Delta\omega} * \hat{F}(\omega)\right]_{\omega=j\,\Delta\omega} \\
&= \frac{1}{\sqrt{2\pi}}\left\{\left[e^{-i\omega t_o}\,\mathrm{Sa}\left(\frac{2\pi\omega}{2\,\Delta\omega}\right)\right] * \hat{F}(\omega)\right\}_{\omega=j\,\Delta\omega} \\
&= \frac{1}{\sqrt{2\pi}}\left\{\int_{-\infty}^{\infty} e^{-i(\omega-\eta)t_o}\,\mathrm{Sa}\left[\frac{2\pi(\omega-\eta)}{2\,\Delta\omega}\right]\hat{F}(\eta)\,d\eta\right\}_{\omega=j\,\Delta\omega} \\
&= \frac{e^{-ij\,\Delta\omega\,t_o}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\eta t_o}\,\mathrm{Sa}\left[\frac{2\pi(j\,\Delta\omega-\eta)}{2\,\Delta\omega}\right]\hat{F}(\eta)\,d\eta.
\end{aligned}
$$

Transforming (19.5),

$$
\hat{F}(\omega) = \left[\sum_{k=-\infty}^{\infty}\delta(\omega - k\,\Delta\omega)\right] F(\omega)\,\Delta\omega. \tag{19.7}
$$

[4]G. Andrew Walls has kindly provided the technique at the author's request. Walls' original is [172].

Using (19.7) against the last expression for a_j,

$$
\begin{aligned}
a_j &= \frac{e^{-ij\,\Delta\omega\,t_o}\,\Delta\omega}{\sqrt{2\pi}} \\
&\qquad \times \int_{-\infty}^{\infty} e^{i\eta t_o}\,\mathrm{Sa}\left[\frac{2\pi(j\,\Delta\omega-\eta)}{2\,\Delta\omega}\right]\left[\sum_{k=-\infty}^{\infty}\delta(\eta-k\,\Delta\omega)\right]F(\eta)\,d\eta \\
&= \frac{e^{-ij\,\Delta\omega\,t_o}\,\Delta\omega}{\sqrt{2\pi}} \\
&\qquad \times \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\eta t_o}\,\mathrm{Sa}\left[\frac{2\pi(j\,\Delta\omega-\eta)}{2\,\Delta\omega}\right][\delta(\eta-k\,\Delta\omega)]\,F(\eta)\,d\eta.
\end{aligned}
$$

The integral might not converge but if it does, sifting,

$$
\begin{aligned}
a_j &= \frac{e^{-ij\,\Delta\omega\,t_o}\,\Delta\omega}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty} e^{ik\,\Delta\omega\,t_o}\,\mathrm{Sa}\left[\frac{2\pi(j\,\Delta\omega-k\,\Delta\omega)}{2\,\Delta\omega}\right]F(k\,\Delta\omega) \\
&= \frac{\Delta\omega}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty} e^{i(k-j)\,\Delta\omega\,t_o}\,\mathrm{Sa}[\pi(j-k)]F(k\,\Delta\omega).
\end{aligned}
$$

The indices j and k being integers,

$$
e^{i(k-j)\,\Delta\omega\,t_o}\,\mathrm{Sa}[\pi(j-k)] = \begin{cases} 1 & \text{if } j=k, \\ 0 & \text{otherwise,} \end{cases}
$$

by which the last expression for a_j implies (19.4).

19.2 Constructive functions

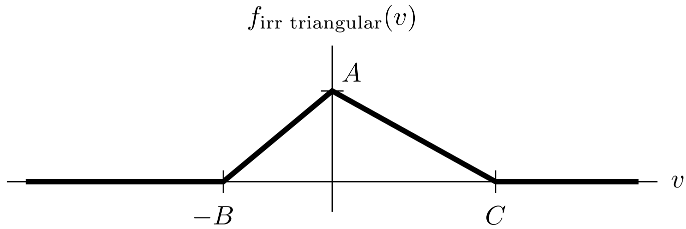
Section 18.1.5 has introduced the approximation and construction, and subsequent transformation, of an arbitrary pulse. That subsection wants evenly spaced samples, though. What if the samples are unevenly spaced?

This section develops constructive functions to represent among others samples that are unevenly spaced.

19.2.1 The irregular triangular pulse

The Fourier transform of the irregular triangular pulse of Fig. 19.2, in which $B > 0$ and $C > 0$, is by

Figure 19.2: An irregular triangular pulse.



$$
\begin{aligned}
\frac{\left(\sqrt{2\pi}\right) F_{\text{irr triangular}}(v)}{A} &= \int_{-B}^{0} e^{-iv\theta}\left(1+\frac{\theta}{B}\right)d\theta + \int_{0}^{C} e^{-iv\theta}\left(1-\frac{\theta}{C}\right)d\theta \\
&= \left[e^{-iv\theta}\left(-\frac{1}{iv}-\frac{\theta}{iBv}+\frac{1}{Bv^2}\right)\right]_{\theta=-B}^{0} \\
&\qquad + \left[e^{-iv\theta}\left(-\frac{1}{iv}+\frac{\theta}{iCv}-\frac{1}{Cv^2}\right)\right]_{\theta=0}^{C} \\
&= \left[\frac{1-e^{iBv}}{Bv^2}-\frac{1}{iv}\right]+\left[\frac{1-e^{-iCv}}{Cv^2}+\frac{1}{iv}\right] \\
&= \frac{1-e^{iBv}}{Bv^2}+\frac{1-e^{-iCv}}{Cv^2}.
\end{aligned}
$$

Thus,

$$
f_{\text{irr triangular}}(v) \xrightarrow{\mathscr{F}} \frac{A}{\sqrt{2\pi}}\left(\frac{1-e^{iBv}}{Bv^2}+\frac{1-e^{-iCv}}{Cv^2}\right), \quad B>0,\ C>0. \tag{19.8}
$$

As v vanishes, $1-e^{\alpha v} \to -\alpha v - \alpha^2 v^2/2 - \cdots$, so

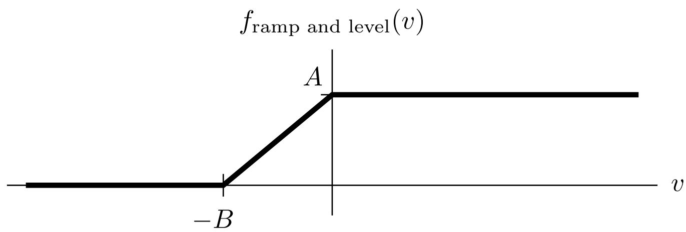
$$
\begin{aligned}
\lim_{v\to 0} F_{\text{irr triangular}}(v) = \lim_{v\to 0} \frac{A}{\sqrt{2\pi}}\Bigg(&\frac{-iBv + B^2v^2/2+\cdots}{Bv^2} \\
&+\frac{iCv + C^2v^2/2+\cdots}{Cv^2}\Bigg),
\end{aligned}
$$

which implies that

$$
F_{\text{irr triangular}}(0) = \frac{(A)(B+C)}{2\sqrt{2\pi}}, \tag{19.9}
$$

a result that matches the result direct application of (18.4) at $v=0$ yields.

Figure 19.3: A ramp and level.



19.2.2 The ramp and level (failing attempt)

This book like other mathematical books seldom explores failing attempts, successful attempts wanting pages enough. Nevertheless, the applied mathematician in practice, scratching paper with his pencil, spends so much time in failing attempts that the occasional review of a failing attempt can be instructive, as follows.

If $C \to \infty$ in Fig. 19.2 then the function of Fig. 19.3 emerges, a function we shall call the *ramp and level.*

The function's Fourier transform would be hard to compute directly but, in the limit, one can carefully extract the transform from (19.8) by letting $C = 1/\epsilon$—the $\epsilon \to 0^+$ being a positive infinitesimal—after which (19.8) has that

$$f_{\text{ramp and level}}(v) \xrightarrow{\mathscr{F}} \lim_{\epsilon \to 0^+} \frac{A}{\sqrt{2\pi}} \left[\frac{1 - e^{iBv}}{Bv^2} + \frac{(\epsilon)\left(1 - e^{-iv/\epsilon}\right)}{v^2} \right].$$

That is,

$$f_{\text{ramp and level}}(v) \xrightarrow{\mathscr{F}} \frac{(A)\left(1 - e^{iBv}\right)}{\left(\sqrt{2\pi}\right) Bv^2},\ v \neq 0, \tag{19.10}$$

in which the condition on the right that $v \neq 0$ does not here forbid the v on the *left* from vanishing.

So far, so good. However, the reasoning § 19.2.1 has used to reach (19.9) for vanishing v is insufficiently precise for the present problem. The trouble is that, in § 19.2.1, only v vanished, whereas now $1/C$ vanishes also. One must ask, is the product Cv to vanish? And if it is not, then is the inverse product $1/Cv$ alternately to vanish? We must decide.

Noticing the exponential factor e^{-iCv} in (19.8), we might try letting the product Cv vanish, defining the finite $\rho \equiv v/\epsilon^2$—where by *finite* we here mean that ρ is neither infinite nor infinitesimal (nor zero), and therefore that v is infinitesimal on the order of ϵ^2. Having thus decided and thus defined, we make the unnumbered equation preceding (19.10) to be

$$\begin{aligned}
&\lim_{\epsilon\to 0^+} F_{\text{ramp and level}}(\epsilon^2\rho) \\
&\quad= \lim_{\epsilon\to 0^+} \frac{A}{\sqrt{2\pi}}\left(\frac{1-e^{iB\epsilon^2\rho}}{B\epsilon^4\rho^2} + \frac{1-e^{-i\epsilon\rho}}{\epsilon^3\rho^2}\right) \\
&\quad= \lim_{\epsilon\to 0^+} \frac{A}{\sqrt{2\pi}}\left(\frac{-iB\epsilon^2\rho + B^2\epsilon^4\rho^2/2+\cdots}{B\epsilon^4\rho^2}\right. \\
&\qquad\qquad\left. + \frac{i\epsilon\rho + \epsilon^2\rho^2/2 - i\epsilon^3\rho^3/6+\cdots}{\epsilon^3\rho^2}\right) \\
&\quad= \lim_{\epsilon\to 0^+} \frac{A}{\sqrt{2\pi}}\left(\frac{1}{2\epsilon} + \frac{3B-i\rho}{6}\right),
\end{aligned}$$

a result which might be interesting but which is not obviously helpful.

The technique used—having already prospered in §§ 5.4, 9.7 and 9.10 among others—remains reasonable, so we shall retain the technique in our toolbox as it were for future use. Notwithstanding, even a reasonable technique can fail to solve some particular problem. Whether *this* technique could with extra effort eventually solve the present problem is not a question we will further pursue (for example, one could alternately try letting $1/Cv$ vanish instead of Cv). We will merely rather observe that, in the present instance, the technique does not seem to have prospered yet.

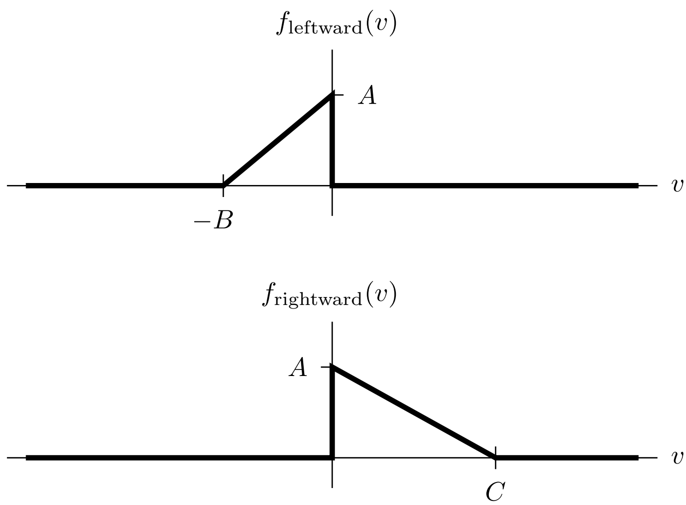
Let us shelve the problem for the moment. We shall retrieve it from the shelf, working it via a different technique, in § 19.2.5.

19.2.3 The right-triangular pulse

If $B = 0$ or $C = 0$ (but not both) in the irregular triangular pulse of § 19.2.1 and Fig. 19.2, then one of the *right-triangular pulses* of Fig. 19.4 results. Depending on whether the pulse consists of the leftward or the rightward triangle, the pulse's Fourier transform is computed as in the earlier subsections to be

$$\begin{aligned}
f_{\text{leftward}}(v) &\xrightarrow{\mathscr{F}} \frac{A}{\sqrt{2\pi}}\left(\frac{1-e^{iBv}}{Bv^2} - \frac{1}{iv}\right), \\
f_{\text{rightward}}(v) &\xrightarrow{\mathscr{F}} \frac{A}{\sqrt{2\pi}}\left(\frac{1-e^{-iCv}}{Cv^2} + \frac{1}{iv}\right),
\end{aligned} \tag{19.11}$$

Figure 19.4: Right-triangular pulses.



with

$$\begin{aligned} F_{\mathrm{leftward}}(0) &= \frac{AB}{2\sqrt{2\pi}}, \\ F_{\mathrm{rightward}}(0) &= \frac{AC}{2\sqrt{2\pi}}. \end{aligned} \tag{19.12}$$

The transform's slow decay, going only on the order of $1/v$, may be worth noticing, incidentally—the slow decay presumably owing[5] to the pulse's discontinuity.

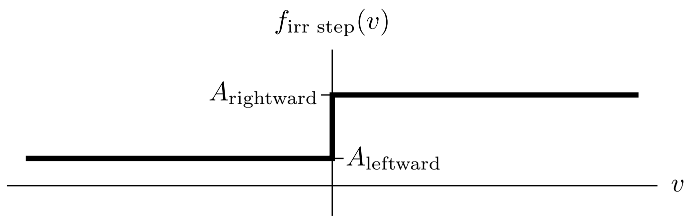
19.2.4 The irregular step

Section 18.3.1 has taken the Fourier transform (18.49) of Heaviside's unit step. Scaling Heaviside vertically by a factor of $A_{\mathrm{rightward}} - A_{\mathrm{leftward}}$ and then, using (18.24), shifting the scaled Heaviside vertically by an offset of A_{leftward} yields the transform pair

$$f_{\mathrm{irr\ step}}(v) \xrightarrow{\mathscr{F}} \frac{A_{\mathrm{rightward}} - A_{\mathrm{leftward}}}{\left(\sqrt{2\pi}\right) iv} + \frac{(A_{\mathrm{rightward}} + A_{\mathrm{leftward}})\sqrt{2\pi}}{2}\delta(v) \tag{19.13}$$

[5]This book will not attempt a general proof of the point.

Figure 19.5: An irregular step.



of the irregular step of Fig. 19.5.

19.2.5 The ramp and level

Sections 19.2.3 and 19.2.4 have drawn their results from a lengthy sequence of applications-level logic, running through chapters 18 and 19, of abstruse Fourier mathematics. Preceding §§ 19.2.3 and 19.2.4 however was § 19.2.2 which, as you will recall, has left one unsolved problem on the shelf. The unsolved problem is the ramp and level, Fig. 19.3.

Superimposing the leftward right-triangular pulse of § 19.2.3 upon the irregular step of § 19.2.4, the latter with $A_{\text{leftward}} = 0$ and $A_{\text{rightward}} = A$, now solves the shelved problem immediately:

$$f_{\text{ramp and level}} \xrightarrow{\mathscr{F}} \frac{A}{\sqrt{2\pi}} \left[\frac{1 - e^{iBv}}{Bv^2} + \pi\delta(v) \right]. \tag{19.14}$$

As well as the ramp and level, one could likewise define a *level and ramp,* the level and ramp mirroring the ramp and level, its level extending not to the right but to the left as in Fig. 19.6. Its transform is evidently

$$f_{\text{level and ramp}} \xrightarrow{\mathscr{F}} \frac{A}{\sqrt{2\pi}} \left[\frac{1 - e^{-iCv}}{Cv^2} + \pi\delta(v) \right]. \tag{19.15}$$

19.2.6 Construction

Section 18.1.5 has employed regular nonanalytic pulses of § 17.3 to approximate a more or less arbitrary function, but the technique of § 18.1.5 remains limited in that it

- cannot track a discontinuity or other infinite slope,

Figure 19.6: A level and ramp.

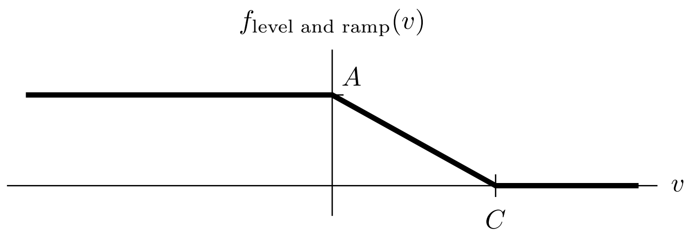


Table 19.1: A sampled function (example).

t	$f(t)$
-50.0	0.00
-10.0	0.15
-2.0	0.92
0.0	1.10
0.5	1.08
4.0	0.60
50.0	0.06

- requires uniform sampling, and

- lacks a practical way to model a function's leftward or rightward tail when, as in Figs. 19.3, 19.5 and 19.6, such a tail exists, $\lim_{v\to\pm\infty} f(v)$ being nonzero.

This section's constructive functions are less limited. An example to illustrate their use follows.

Consider an imprecisely known function $f(t)$ that, though imperfectly, were nevertheless experimentally measurable. Suppose that the decimal samples of Table 19.1 were observed. One might then approximate the function as

$$\begin{aligned}
f(t) \approx{}& \\
& f_{\text{irr triangular},A=0.15,B=(-10.0)-(-50.0),C=(\;\;-2.0)-(-10.0)}[t-(-10.0)] \\
&+ f_{\text{irr triangular},A=0.92,B=(\;\;-2.0)-(-10.0),C=(\;\;\;\;0.0)-(\;\;-2.0)}[t-(\;\;-2.0)] \\
&+ f_{\text{irr triangular},A=1.10,B=(\;\;\;\;0.0)-(\;\;-2.0),C=(\;\;\;\;0.5)-(\;\;\;\;0.0)}[t-(\;\;\;\;0.0)] \\
&+ f_{\text{irr triangular},A=1.08,B=(\;\;\;\;0.5)-(\;\;\;\;0.0),C=(\;\;\;\;4.0)-(\;\;\;\;0.5)}[t-(\;\;\;\;0.5)] \\
&+ f_{\text{irr triangular},A=0.60,B=(\;\;\;\;4.0)-(\;\;\;\;0.5),C=(\;\;50.0)-(\;\;\;\;4.0)}[t-(\;\;\;\;4.0)] \\
&+ f_{\text{ramp and level},A=0.06,B=(50.0)-(4.0)}[t-(50.0)],
\end{aligned}$$

constructing the approximation from this section's various constructive functions, each of which is Fourier transformable as the section has shown.

Note the manner in which the example's several triangular pulses overlap. Observe that the example's approximation exactly fits its experimental samples, interpolating between along straight lines. Such techniques are generally useful.

Variations and refinements are possible, like using an irregular raised cosine (extending § 17.3) in place of the irregular triangle, but this book will pursue the matter no further.

19.3 The Laplace transform

Fourier straightforwardly transforms pulses like those of Figs. 18.1 (page 615) and 17.3 (page 584) but stumbles on time-unlimited functions like $f(t) = \cos \omega_o t$ or even the ridiculously simple $f(t) = 1$. Only by the clever, indirect techniques of §§ 18.2 and 18.3 has Fourier been able to transform such functions at all. Fourier's clever, indirect techniques are valid and even interesting but can still prompt a mathematician to wonder whether a simpler alternative to Fourier did not exist.

At the sometimes acceptable cost of omitting one of the Fourier integral's two tails,[6] the *Laplace transform*

$$F(s) = \mathscr{L}\{f(t)\} \equiv \int_{0^-}^{\infty} e^{-st} f(t)\, dt \tag{19.16}$$

offers such an alternative. Here, $s = i\omega$ is the transform variable and, when s is purely imaginary, the Laplace transform is very like the Fourier; but

[6]There has been invented a version of the Laplace transform which omits no tail [83, chapter 3]. This book does not treat it.

Laplace's advantage lies in that it encourages the use of a complex s, usually with a negative real part, which in (19.16)'s integrand tends to suppress even the tail not omitted, thus effectively converting a time-unlimited function to an integrable pulse—and Laplace does all this without resort to indirect techniques.[7]

The lower limit 0^- of $\int_{0^-}^{\infty}$ integration means that the integration includes $t = 0$. In other words, $\int_{0^-}^{\infty} \delta(t)\,dt = 1$ whereas $\int_{0^+}^{\infty} \delta(t)\,dt = 0$.

Without resort to clever, indirect techniques, Laplace's (19.16) can transform

$$1 \xrightarrow{\mathscr{L}} \frac{1}{s},$$

listed among other pairs by Table 19.3. Laplace transform *properties,* some of which are derived in the same easy way, are listed in Table 19.2. Further Laplace properties, also listed in the table, want some technique to derive, for instance the differentiation property, which comes by

$$\begin{aligned}
\mathscr{L}\left\{\frac{d}{dt}f(t)\right\} &= \int_{0^-}^{\infty} e^{-st}\frac{d}{dt}f(t)\,dt = \int_{t=0^-}^{\infty} e^{-st}\,d\left[f(t)\right] \\
&= e^{-st}f(t)\big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} e^{-st}f(t)\,dt \\
&= -f(0^-) + sF(s)
\end{aligned}$$

via the by-parts method of § 9.4. The integration property merely reverses the differentiation property on the function $g(t) \equiv \int_{0^-}^{t} f(\tau)\,d\tau$, for which $g(0^-) = 0$, filling the "?" with $F(s)/s$ in this pattern:

$$\begin{aligned}
g(t) &= \int_{0^-}^{t} f(\tau)\,d\tau &\xrightarrow{\mathscr{F}}\quad & G(s) &&= \,?\ ; \\
\frac{dg}{dt} &= \qquad f(t) &\xrightarrow{\mathscr{F}}\quad & 0 + sG(s) = F(s). &&
\end{aligned}$$

The ramping property comes by differentiating and negating (19.16) as

$$-\frac{d}{ds}F(s) = -\frac{d}{ds}\int_{0^-}^{\infty} e^{-st}f(t)\,dt = \int_{0^-}^{\infty} e^{-st}[tf(t)]\,dt = \mathscr{L}\{tf(t)\}.$$

Higher-order properties come by repeated application. The convolution property comes as it did in § 18.2.7, beginning

$$\begin{aligned}
&\mathscr{L}\left\{[u_1(t)h(t)] * [u_1(t)f(t)]\right\} \\
&\quad = \int_{-\infty}^{\infty} e^{-st}\int_{-\infty}^{\infty} u_1\left(\frac{t}{2}-\psi\right)h\left(\frac{t}{2}-\psi\right)u_1\left(\frac{t}{2}+\psi\right)f\left(\frac{t}{2}+\psi\right)d\psi\,dt,
\end{aligned}$$

[7] [83, chapter 3][129, chapter 7][89, chapter 19]

Table 19.2: Properties of the Laplace transform.

$$
\begin{aligned}
u_1(t-t_o)f(t-t_o) &\overset{\mathscr{L}}{\rightarrow} e^{-st_o}F(s),\ t_o \ge 0 \\
e^{-at}f(t) &\overset{\mathscr{L}}{\rightarrow} F(s+a) \\
Af(\alpha t) &\overset{\mathscr{L}}{\rightarrow} \frac{A}{\alpha}F\left(\frac{s}{\alpha}\right),\ \Im(\alpha)=0,\ \Re(\alpha)>0 \\
A_1f_1(t)+A_2f_2(t) &\overset{\mathscr{L}}{\rightarrow} A_1F_1(t)+A_2F_2(t) \\
\frac{d}{dt}f(t) &\overset{\mathscr{L}}{\rightarrow} sF(s)-f(0^-) \\
\frac{d^2}{dt^2}f(t) &\overset{\mathscr{L}}{\rightarrow} s^2F(s)-sf(0^-)-\left[\frac{df}{dt}\right]_{t=0^-} \\
\frac{d^n}{dt^n}f(t) &\overset{\mathscr{L}}{\rightarrow} s^nF(s)-\sum_{k=0}^{n-1}\left\{s^k\left[\frac{d^{n-1-k}}{dt^{n-1-k}}f(t)\right]_{t=0^-}\right\} \\
\int_{0^-}^{t} f(\tau)\,d\tau &\overset{\mathscr{L}}{\rightarrow} \frac{F(s)}{s} \\
tf(t) &\overset{\mathscr{L}}{\rightarrow} -\frac{d}{ds}F(s) \\
t^nf(t) &\overset{\mathscr{L}}{\rightarrow} (-)^n\frac{d^n}{ds^n}F(s) \\
[u_1(t)h(t)]*[u_1(t)f(t)] &\overset{\mathscr{L}}{\rightarrow} H(s)F(s)
\end{aligned}
$$

As in § 18.2.7, here also we change $\phi \leftarrow t/2+\psi$ and $\mu \leftarrow \phi-2\psi$, eventually reaching the form

$$
\begin{aligned}
&\mathscr{L}\left\{[u_1(t)h(t)]*[u_1(t)f(t)]\right\} \\
&\quad = \left[\int_{-\infty}^{\infty} e^{-s\mu}u_1(\mu)h(\mu)\,d\mu\right]\left[\int_{-\infty}^{\infty} e^{-s\phi}u_1(\phi)f(\phi)\,d\phi\right],
\end{aligned}
$$

after which we take advantage of the presence of Heaviside's unit step $u_1(\cdot)$ of (7.22) to curtail each integration to begin at 0^- rather than at $-\infty$, thus completing the convolution property's proof.

Splitting the sine and cosine functions into their complex exponential components according to Table 5.1, application of Laplace's definition (19.16) to each component yields Table 19.3's sine and cosine pairs. The pair $t \overset{\mathscr{L}}{\rightarrow} 1/s^2$ of the latter table comes by application of the property that

Table 19.3: Laplace transform pairs.

$$
\begin{array}{rclrcl}
 & & & e^{-at}t\sin\omega_o t & \stackrel{\mathscr{L}}{\to} & \dfrac{2(s+a)\omega_o}{[(s+a)^2+\omega_o^2]^2} \\
\delta(t) & \stackrel{\mathscr{L}}{\to} & 1 & e^{-at}t\cos\omega_o t & \stackrel{\mathscr{L}}{\to} & \dfrac{(s+a)^2-\omega_o^2}{[(s+a)^2+\omega_o^2]^2} \\
1 & \stackrel{\mathscr{L}}{\to} & \dfrac{1}{s} & e^{-at} & \stackrel{\mathscr{L}}{\to} & \dfrac{1}{s+a} \\
t & \stackrel{\mathscr{L}}{\to} & \dfrac{1}{s^2} & e^{-at}t & \stackrel{\mathscr{L}}{\to} & \dfrac{1}{(s+a)^2} \\
t^n & \stackrel{\mathscr{L}}{\to} & \dfrac{n!}{s^{n+1}} & e^{-at}t^n & \stackrel{\mathscr{L}}{\to} & \dfrac{n!}{(s+a)^{n+1}} \\
\sin\omega_o t & \stackrel{\mathscr{L}}{\to} & \dfrac{\omega_o}{s^2+\omega_o^2} & e^{-at}\sin\omega_o t & \stackrel{\mathscr{L}}{\to} & \dfrac{\omega_o}{(s+a)^2+\omega_o^2} \\
\cos\omega_o t & \stackrel{\mathscr{L}}{\to} & \dfrac{s}{s^2+\omega_o^2} & e^{-at}\cos\omega_o t & \stackrel{\mathscr{L}}{\to} & \dfrac{s+a}{(s+a)^2+\omega_o^2}
\end{array}
$$

$tf(t) \stackrel{\mathscr{L}}{\to} -(d/ds)F(s)$ to the pair $1 \stackrel{\mathscr{L}}{\to} 1/s$, and the pair $t^n \stackrel{\mathscr{L}}{\to} n!/s^{n+1}$ comes by repeated application of the same property. The pairs transforming $e^{-at}t\sin\omega_o t$, $e^{-at}t\cos\omega_o t$, $e^{-at}t$ and $e^{-at}t^n$ come similarly.

During application of either Table 19.2 or Table 19.3, a may be, and s usually is, complex; but α, ω_o and t are normally real.

19.4 Solving differential equations by Laplace

The Laplace transform is curious, but Fourier is admittedly more straightforward even if it is harder to analyze. Besides being more straightforward, Fourier brings an inverse transformation formula (18.5) whereas Laplace does not.[8]

Laplace excels Fourier however in its property of Table 19.2 that $(d/dt)f(t) \stackrel{\mathscr{L}}{\to} sF(s) - f(0^-)$. Fourier's corresponding property of Table 18.1 lacks the $f(0^-)$, an initial condition.

[8] Actually, formally, Laplace does support an inverse transformation formula, $u_1(t)f(t) = (1/i2\pi)\int_{-i\infty}^{i\infty} e^{st}F(s)\,ds$, but to apply this inverse requires contour integration [129, eqn. 7.2]. The writer has no experience with it. We'll not use it. It comes of changing $s \leftarrow i\omega$ in (18.1).

To see why this matters, consider for example the linear differential equation[9,10]

$$\begin{aligned} \frac{d^2}{dt^2}f(t) + 4\frac{d}{dt}f(t) + 3f(t) &= e^{-2t}, \quad t \geq 0; \\ f(t)|_{t=0^-} &= 1; \\ \left.\frac{d}{dt}f(t)\right|_{t=0^-} &= 2. \end{aligned}$$

Applying the properties of Table 19.2 and transforms of Table 19.3, term by term, yields the transformed equation

$$\begin{aligned} &\left\{ s^2F(s) - s\Big[f(t)\Big]_{t=0^-} - \left[\frac{d}{dt}f(t)\right]_{t=0^-} \right\} \\ &\qquad + 4\left\{ sF(s) - \Big[f(t)\Big]_{t=0^-} \right\} + 3F(s) = \frac{1}{s+2}. \end{aligned}$$

That is,

$$(s^2 + 4s + 3)F(s) - (s+4)\Big[f(t)\Big]_{t=0^-} - \left[\frac{d}{dt}f(t)\right]_{t=0^-} = \frac{1}{s+2}.$$

Applying the known initial conditions,

$$(s^2 + 4s + 3)F(s) - (s+4)[1] - [2] = \frac{1}{s+2}.$$

Combining like terms,

$$(s^2 + 4s + 3)F(s) - (s+6) = \frac{1}{s+2}.$$

Multiplying by $s+2$ and rearranging,

$$(s+2)(s^2 + 4s + 3)F(s) = s^2 + 8s + 0\mathrm{xD}.$$

[9] [89, Example 19.31]

[10] It is enlightening to study the same differential equation in *state-space* style [129, chapter 8],

$$\frac{d}{dt}\mathbf{f}(t) = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\mathbf{f}(t) + \begin{bmatrix} 0 \\ e^{-2t} \end{bmatrix}, \quad \mathbf{f}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

where

$$\mathbf{f}(t) \equiv \begin{bmatrix} 1 \\ d/dt \end{bmatrix} f(t).$$

The effort required to assimilate the notation rewards the student with significant insight into the manner in which initial conditions—here symbolized $\mathbf{f}(0)$—determine a system's subsequent evolution.

Isolating the heretofore unknown frequency-domain function $F(s)$,

$$F(s) = \frac{s^2 + 8s + \mathrm{0xD}}{(s+2)(s^2+4s+3)}.$$

Factoring the denominator,

$$F(s) = \frac{s^2 + 8s + \mathrm{0xD}}{(s+1)(s+2)(s+3)}.$$

Expanding in partial fractions (this step being the key to the whole technique: see § 9.7),

$$F(s) = \frac{3}{s+1} - \frac{1}{s+2} - \frac{1}{s+3}.$$

Though we lack an inverse transformation formula, it seems that we do not need one because—having split the frequency-domain equation into such simple terms—we can just look up the inverse transformation in Table 19.3, term by term. The time-domain solution

$$f(t) = 3e^{-t} - e^{-2t} - e^{-3t}$$

results. One can verify the solution by substituting it back into differential equation.

Laplace can solve many linear differential equations in this way.

19.5 Initial and final values by Laplace

The method of § 19.4 though effective is sometimes too much work, when all one wants to know are the initial and/or final values of a function $f(t)$, when one is uninterested in the details between. The Laplace transform's *initial-* and *final-value theorems,*

$$\begin{aligned} f(0^+) &= \lim_{s\to\infty} sF(s), \\ \lim_{t\to\infty} f(t) &= \lim_{s\to 0} sF(s), \end{aligned} \tag{19.17}$$

meet this want.

One derives the initial-value theorem via the steps

$$
\begin{aligned}
&\lim_{s\to\infty} sF(s) - f(0^-) \\
&\quad= \lim_{s\to\infty} \mathscr{L}\left\{\frac{d}{dt}f(t)\right\} \\
&\quad= \lim_{s\to\infty} \int_{0^-}^{\infty} e^{-st}\frac{d}{dt}f(t)\,dt \\
&\quad= \lim_{\epsilon\to 0^+} \left\{\int_{0^-}^{\infty} e^{-st}\frac{d}{dt}f(t)\,dt\right\}_{s=1/\epsilon^2} \\
&\quad= \lim_{\epsilon\to 0^+} \left\{\int_{0^-}^{0^+} e^{-st}\frac{d}{dt}f(t)\,dt + \int_{0^+}^{\epsilon} e^{-st}\frac{d}{dt}f(t)\,dt \right. \\
&\qquad\qquad\qquad \left. + \int_{\epsilon}^{\infty} e^{-st}\frac{d}{dt}f(t)\,dt\right\}_{s=1/\epsilon^2} \\
&\quad= \lim_{\epsilon\to 0^+} \left\{\int_{0^-}^{0^+} \frac{d}{dt}f(t)\,dt + \left[\frac{d}{dt}f(t)\right]_{t=0^+} \int_{0^+}^{\epsilon} e^{-st}\,dt\right\}_{s=1/\epsilon^2} \\
&\quad= \int_{0^-}^{0^+} \frac{d}{dt}f(t)\,dt = f(0^+) - f(0^-),
\end{aligned}
$$

which invoke the time-differentiation property of Table 19.2 and the last of which implies (19.17)'s first line. For the final value, one begins

$$
\begin{aligned}
\lim_{s\to 0} sF(s) - f(0^-) &= \lim_{s\to 0} \mathscr{L}\left\{\frac{d}{dt}f(t)\right\} \\
&= \lim_{s\to 0} \int_{0^-}^{\infty} e^{-st}\frac{d}{dt}f(t)\,dt \\
&= \int_{0^-}^{\infty} \frac{d}{dt}f(t)\,dt = \lim_{t\to\infty} f(t) - f(0^-),
\end{aligned}
$$

and (19.17)'s second line follows immediately.[11]

19.6 The spatial Fourier transform

In the study of wave mechanics, physicists and engineers sometimes elaborate the Fourier transform's kernel $e^{iv\theta}$ or $e^{i\omega t}$, or by whichever pair of letters is let to represent the complementary variables of transformation, into the

[11] [129, § 7.5]

more general, spatiotemporal phase factor[12] $e^{i(\pm\omega t \mp \mathbf{k}\cdot\mathbf{r})}$; where $\mathbf{k}$ and $\mathbf{r}$ are three-dimensional geometrical vectors and $\mathbf{r}$ in particular represents a position in space. To review the general interpretation and use of such a factor lies beyond the book's scope but the factor's very form,

$$e^{i(\pm\omega t \mp \mathbf{k}\cdot\mathbf{r})} = e^{i(\pm\omega t \mp k_x x \mp k_y y \mp k_z z)},$$

suggests Fourier transformation with respect not merely to time but also to space. There results the *spatial Fourier transform*

$$\begin{aligned} F(\mathbf{k}) &= \frac{1}{(2\pi)^{3/2}} \int_V e^{+i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r})\,d\mathbf{r}, \\ f(\mathbf{r}) &= \frac{1}{(2\pi)^{3/2}} \int_V e^{-i\mathbf{k}\cdot\mathbf{r}} F(\mathbf{k})\,d\mathbf{k}, \end{aligned} \tag{19.18}$$

analogous to (18.5) but cubing the $1/\sqrt{2\pi}$ scale factor for the triple integration and reversing the sign of the kernel's exponent. The transform variable $\mathbf{k}$, analogous to ω, is a *spatial frequency,* also for other reasons called a *propagation vector.*

Nothing prevents one from extending (19.18) to four dimensions, including a fourth integration to convert time t to temporal frequency ω while also converting position $\mathbf{r}$ to spatial frequency $\mathbf{k}$. On the other hand, one can restrict it to two dimensions or even one. Thus, various plausibly useful Fourier transforms include

$$\begin{aligned} F(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt, \\ F(k_z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{+ik_z z} f(z)\,dz, \\ F(\mathbf{k}_\rho) &= \frac{1}{2\pi} \int_S e^{+i\mathbf{k}_\rho\cdot\boldsymbol{\rho}} f(\boldsymbol{\rho})\,d\boldsymbol{\rho}, \\ F(\mathbf{k}) &= \frac{1}{(2\pi)^{3/2}} \int_V e^{+i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r})\,d\mathbf{r}, \\ F(\mathbf{k},\omega) &= \frac{1}{(2\pi)^2} \int_V \int_{-\infty}^{\infty} e^{i(-\omega t+\mathbf{k}\cdot\mathbf{r})} f(\mathbf{r},t)\,dt\,d\mathbf{r}, \end{aligned}$$

among others.

[12] The choice of sign here is a matter of convention, which differs by discipline. This book tends to reflect its author's preference for $f(\mathbf{r},t) \sim \int e^{i(+\omega t-\mathbf{k}\cdot\mathbf{r})} F(\mathbf{k},\omega)\,d\omega\,d\mathbf{k}$, convenient in electrical modeling but slightly less convenient in quantum-mechanical work.

19.7 The Fourier transform in cyclic frequencies

Many applications prefer[13] a clever alternate definition of the Fourier transform,[14]

$$\begin{aligned} \Phi(\nu) &\equiv \int_{-\infty}^{\infty} e^{-i2\pi\nu t} f(t)\,dt, && (19.19)\\ \nu &\equiv \frac{\omega}{2\pi}, && (19.20) \end{aligned}$$

instead of (18.5). In (19.19) and (19.20), cyclic frequency ν appears rather than angular frequency ω. Comparing against (18.5),

$$\Phi(\nu) = \left(\sqrt{2\pi}\right) F(\omega), \tag{19.21}$$

such that, for example, referring to Table 18.4, if $f(t) = \Lambda(t)$, then $\Phi(t) = \mathrm{Sa}^2(\pi\nu)$—or, if you prefer, letting this section use the symbol $\rightarrow$ to announce the cyclic transform (19.19), $f(t) \rightarrow \mathrm{Sa}^2(\pi\nu)$.

Substituting (19.20) and (19.21) into (18.5), the inverse cyclic transform is

$$f(t) = \int_{-\infty}^{\infty} e^{i2\pi\nu t}\Phi(\nu)\,d\nu, \tag{19.22}$$

an elegant result. Even more elegant are the alternate forms of the convolution properties of Table 18.3. Defining $s(f) \equiv h(t) * f(t)$ and letting $\Sigma(\nu)$, $\mathrm{X}(\nu)$ and $\Phi(\nu)$ respectively be the cyclic alternatives to $S(\omega)$, $H(\omega)$ and $F(\omega)$, we have that $\Sigma(\nu) = (\sqrt{2\pi})S(\omega) = 2\pi H(\omega)F(\omega) = \mathrm{X}(\nu)\Phi(\nu)$. Calculating in such a manner,

$$\begin{aligned} h(t) * f(t) &\rightarrow \mathrm{X}(\nu)\Phi(\nu),\\ h(t)f(t) &\rightarrow \mathrm{X}(\nu) * \Phi(\nu), \end{aligned} \tag{19.23}$$

[13][37]

[14]The Φ of this section is not the Φ of § 18.2. The two sections use the Greek letter Φ for different purposes.

the latter because

$$\begin{aligned}
h(t)f(t) &\stackrel{\mathscr{F}}{\to} \frac{1}{\sqrt{2\pi}}[H(\omega) * F(\omega)] \\
&\stackrel{\mathscr{F}}{\to} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H\left(\frac{\omega}{2} - \eta\right) F\left(\frac{\omega}{2} + \eta\right) d\eta \\
&\stackrel{\mathscr{F}}{\to} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \mathrm{X}\left(\frac{\nu}{2} - \nu'\right)\right] \left[\frac{1}{\sqrt{2\pi}} \Phi\left(\frac{\nu}{2} + \nu'\right)\right] 2\pi \, d\nu' \\
&\stackrel{\mathscr{F}}{\to} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{X}\left(\frac{\nu}{2} - \nu'\right) \Phi\left(\frac{\nu}{2} + \nu'\right) d\nu' \\
&\stackrel{\mathscr{F}}{\to} \frac{1}{\sqrt{2\pi}}[\mathrm{X}(\nu) * \Phi(\nu)],
\end{aligned}$$

the third line of which has used (19.21) twice and, at its far right, changed $2\pi \, d\nu' \leftarrow d\eta$ according to the proportion (19.20). Just how the oddly written angular Fourier transform pair that results justifies the cyclic Fourier transform pair on (19.23)'s second line might not seem obvious at a glance, but see: ν according to (19.20) is but a scaled version of ω, so the expression $[1/\sqrt{2\pi}][\mathrm{X}(\nu) * \Phi(\nu)]$ is a function of ω just as well as it is a function of ν. Defining the symbol

$$G(\omega) \equiv \frac{1}{\sqrt{2\pi}}[\mathrm{X}(\nu) * \Phi(\nu)],$$

we have the angular transform pair

$$h(t)f(t) \stackrel{\mathscr{F}}{\to} G(\omega).$$

If the symbol $\Gamma(\nu)$ is now introduced to represent the *cyclic* transform corresponding to $G(\omega)$, then

$$h(t)f(t) \to \Gamma(\nu);$$

but then too according to (19.21),

$$\Gamma(\nu) = \left(\sqrt{2\pi}\right) G(\omega).$$

When the expression the symbol $G(\omega)$ has been defined to represent is substituted into the last, the cyclic transform pair on (19.23)'s second line emerges. Thus we see that, so long as one does not mind forgoing the use of angular frequencies, the cyclic alternative brings some practical advantages.

Table 19.4 lists some notable Fourier properties in the cyclic style. Ta-

Table 19.4: Properties of the cyclic Fourier transform.

$$
\begin{array}{rcl}
\Phi(\nu) & = & \left(\sqrt{2\pi}\right) F(\omega) \\
f(v-b) & \to & e^{-ib2\pi v}\Phi(v) \\
e^{iav} f(v) & \to & \Phi\left(v - \dfrac{a}{2\pi}\right) \\
\left(\sqrt{|\alpha|}\right) A f(\alpha v) & \to & \dfrac{A}{\sqrt{|\alpha|}}\Phi\left(\dfrac{v}{\alpha}\right) \\
A f[(\alpha)(v-b)] & \to & \dfrac{A e^{-ib2\pi v}}{|\alpha|}\Phi\left(\dfrac{v}{\alpha}\right) \\
 & & \quad \Im(\alpha) = 0,\ \Re(\alpha) \neq 0 \\
\dfrac{d^n}{dv^n} f(v) & \to & (i2\pi v)^n \Phi(v) \\
(-iv)^n f(v) & \to & \dfrac{d^n}{(2\pi)^n\, dv^n}\Phi(v) \\
 & & \quad n \in \mathbb{Z},\ n \geq 0 \\
\displaystyle\int_{-\infty}^{v} f(\tau)\, d\tau & \to & \dfrac{\Phi(v)}{i2\pi v} + \dfrac{\Phi(0)\delta(v)}{2} \\
\displaystyle\int_{-\infty}^{\infty} h^*(v) f(v)\, dv & = & \displaystyle\int_{-\infty}^{\infty} \mathrm{X}^*(v)\Phi(v)\, dv \\
\displaystyle\int_{-\infty}^{\infty} |f(v)|^2\, dv & = & \displaystyle\int_{-\infty}^{\infty} |\Phi(v)|^2\, dv \\
a_j & = & \Delta\nu\, \Phi(j\, \Delta\nu) \\
h(t) * f(t) & \to & \mathrm{X}(\nu)\Phi(\nu) \\
h(t) f(t) & \to & \mathrm{X}(\nu) * \Phi(\nu) \\
R_{fh}(t) & \to & \mathrm{X}^*(\nu)\Phi(\nu) \\
h^*(t) f(t) & \to & R_{\Phi \mathrm{X}}(\nu) \\
R_{ff}(t) & \to & |\Phi(\nu)|^2
\end{array}
$$

Table 19.5: Cyclic Fourier transform pairs.

$$
\begin{aligned}
1 &\to \delta(v) & \delta(v) &\to 1 \\
\Lambda(v) &\to \mathrm{Sa}^2(\pi v) & u(v) &\to \frac{1}{i2\pi v} + \frac{\delta(v)}{2} \\
\Lambda_r(v) &\to \mathrm{Sa}(\pi r v)\,\mathrm{Sa}(\pi v) & \Pi(v) &\to \mathrm{Sa}\,\pi v \\
\Psi(v) &\to \frac{\mathrm{Sa}\,2\pi v}{[1-(2v)^2]} & \mathscr{F}_{\mathrm{cyc}}\{\Psi(v)\}_{v=\pm 1/2} &= \frac{1}{2} \\
\Psi_r(v) &\to \frac{\cos(r\pi v)\,\mathrm{Sa}(\pi v)}{[1-(2rv)^2]} & \mathscr{F}_{\mathrm{cyc}}\{\Psi_r(v)\}_{rv=\pm 1/2} &= \frac{2\pi\,\mathrm{Sa}(\pi v)}{8} \\
& & &= \frac{2\pi\,\mathrm{Sa}(2\pi/4r)}{8} \\
\mathrm{Sa}^2(v) &\to \pi\Lambda(\pi v) & \mathrm{Sa}(v) &\to \pi\Pi(\pi v)
\end{aligned}
$$

$$
\begin{aligned}
u(v)e^{-av} &\to \frac{1}{a+i2\pi v}, && \Re(a) > 0 \\
u(v)e^{-av}v^n &\to \frac{n!}{(a+i2\pi v)^{n+1}}, && \Re(a) > 0,\ n \in \mathbb{Z},\ n \geq 0 \\
e^{ib2\pi v} &\to \delta(v-b), && \Im(b) = 0 \\
e^{iav} &\to \delta\left(v - \frac{a}{2\pi}\right), && \Im(a) = 0 \\
\sin av &\to \frac{1}{i2}\left[\delta\left(v-\frac{a}{2\pi}\right) - \delta\left(v+\frac{a}{2\pi}\right)\right], && \Im(a) = 0 \\
\cos av &\to \frac{1}{2}\left[\delta\left(v-\frac{a}{2\pi}\right) + \delta\left(v+\frac{a}{2\pi}\right)\right], && \Im(a) = 0 \\
\sqrt{T_1}\sum_{j=-\infty}^{\infty}\delta(v-jT_1) &\to \sqrt{\Delta\nu}\sum_{j=-\infty}^{\infty}\delta(v-j\,\Delta\nu), && \Delta\nu\,T_1 = 1 \\
\Omega\left[\left(\sqrt{2\pi}\right)v\right] &\to \Omega\left[\left(\sqrt{2\pi}\right)v\right]
\end{aligned}
$$

ble 19.5 lists cyclic Fourier transform pairs. The pairs of the latter table are merely the pairs of Tables 18.4 and 18.5 converted according to (19.21) with (where $\delta[v]$ appears on a pair's right) appeal to (7.26). Most of the properties of the former table, too, are converted in the same way from Table 18.1, as for example—observing the difference between the symbols $\stackrel{\mathscr{F}}{\rightarrow}$ and $\rightarrow$, only the latter of which announces the cyclic transform—the angular property that $f(t-b) \stackrel{\mathscr{F}}{\rightarrow} e^{-ib\omega}F(\omega) = [e^{-ib2\pi\nu}][\Phi(\nu)/\sqrt{2\pi}]$, reäpplication of (19.21) to which yields the cyclic property that $f(t-b) \rightarrow e^{-ib2\pi\nu}\Phi(\nu)$; or as the angular property that $e^{iat}f(t) \stackrel{\mathscr{F}}{\rightarrow} F[\omega - a] = \Phi[(\omega - a)/2\pi]/\sqrt{2\pi} = \Phi[\nu - (a/2\pi)]/\sqrt{2\pi}$, reäpplication of (19.21) to which yields the cyclic property that $e^{iat}f(t) \rightarrow \Phi[\nu - (a/2\pi)]$. The various convolution and correlation properties are converted from Table 18.3, instead, by the method this section has earlier shown. The integration property does not easily submit to either conversion technique but can be derived as in § 18.5 using Table 19.5's cyclic transform of $u(v)$.

Whether to prefer the angular or the cyclic style of the Fourier transform is a matter of taste and circumstance. The angular style better comports with familiar expressions like $\sin \omega t$, yet the way the cyclic alternative dispenses with so many stray factors of $\sqrt{2\pi}$ is indeed attractive; and where frequencies are expressed in hertz, the cyclic style may seem the more natural. On the other hand, $\Omega(t) \rightarrow (\sqrt{2\pi})\Omega(\nu)$, whereas $\Omega(t) \stackrel{\mathscr{F}}{\rightarrow} \Omega(\omega)$, so the angular style wins *that* contest; and the angular delivers a more appealing derivative property, as well. Applied mathematicians can put both styles to good use.[15] The book you are reading defaults to the angular style as you see; you can do likewise if you wish.

[15] In some countries including the writer's, engineers have tended to use a third definition of the transform, $F_{\text{engineer}}(\omega) = (\sqrt{2\pi})F(\omega)$. Working with engineers (the writer is one), the writer has had occasion to employ the third definition many times. Engineers are fine people but the writer cannot especially recommend their third definition, which obscures the natural, mutual symmetry of the transform and its inverse.

Chapter 20

Probability

Of all mathematical fields of study, none may be so counterintuitive and yet so widely applied as that of probability—whether as *probability* in the technical term's conventional, restricted meaning or as probability in its expanded or inverted guise as *statistics.*[1] The untrained mind seems to rebel against the concept. Nevertheless, sustained reflection upon the concept gradually reveals a fascinating mathematical landscape.

As calculus is the mathematics of change, so *probability* is the mathematics of uncertainty. If I tell you that my thumb is three inches long, I likely do not mean that it is exactly three inches. I mean that it is about three inches. Quantitatively, I might report the length as 3.0 ± 0.1 inches, thus indicating not only the length but the degree of uncertainty in the length. Probability in its guise as *statistics* is the mathematics which produces, analyzes and interprets such quantities.

More obviously statistical is a report that, say, the average 25-year-old U.S. male is 69 ± 3 inches tall, inferred from actual measurements[2] of some number $N > 1$ of 25-year-old U.S. males. Deep mathematics underlie such a report, for the report implies among other things that a little over two-thirds—$(1/\sqrt{2\pi}) \int_{-1}^{1} \exp(-\tau^2/2)\, d\tau \approx$ 0x0.AEC5, to be precise—of a typical, randomly chosen sample of 25-year-old U.S. males ought to be found

[1] The nomenclature is slightly unfortunate. Were statistics called "inferred probability" or "probabilistic estimation" the name would suggest something like the right taxonomy. Actually, the nomenclature is fine once you know what it means, but on first encounter it provokes otherwise unnecessary footnotes like this one.

Statistics (singular noun) the expanded mathematical discipline—as opposed to the statistics (plural noun) mean and standard deviation of § 20.2—as such lies mostly beyond this book's scope, but the chapter will have at least a little to say about it in § 20.6.

[2] [34, Tables 6 and 12]

to have heights between 66 and 72 inches.

Probability is also met in games of chance and in models of systems which—from the model's point of view—logically resemble games of chance, and in this setting probability is not statistics. The reason it is not is that its mathematics in this case is based not on observation but on a teleological assumption of some kind, often an assumption of symmetry such as that no face of a die or card from a deck ought to turn up more often than another. Entertaining so plausible an assumption, if you should draw three cards at random from a standard 52-card[3] deck, the deck comprising four cards each of thirteen ranks, then there would be some finite probability—which is $(3/51)(2/50) = 1/425$—that the three cards drawn would share the same rank (why?). If I should however shuffle the deck, draw three cards off the top, and look at the three cards without showing them to you, all before inviting you to draw three, then the probability that your three would share the same rank were again $1/425$ (why?). On the other hand, if before you drew I let you *peek* at my three hidden cards, and you saw that my three hidden cards were ace, queen and ten, then this knowledge alone must slightly lower your estimate of the probability that your three would subsequently share the same rank to $(40/49)(3/48)(2/47)+(9/49)(2/48)(1/47) \approx 1/428$ (again, why?).

That the probability should be $1/425$ suggests that one would draw three of the same rank once in 425 tries. That is, were I to shuffle 425 decks and you to draw three cards from each, then for you to draw three of the same rank from just one of the 425 decks would be expected. Nevertheless, despite expectations, you *might* draw three of the same rank from two, three or four decks, or from none at all—so what does a probability of $1/425$ really mean? The answer is that it means something like this: were I to shuffle 425 *million* decks then you would draw three of the same rank from very nearly 1.0 million decks—almost certainly not from as many as 1.1 million nor as few as 0.9 million. It means that the ratio of the number of three-of-the-same-rank events to the number of trials must converge exactly upon $1/425$ as the number of trials tends toward infinity.

See also § 4.2.

If unsure, consider this. Suppose that during six days in an unfamiliar climate, in a place you had never before visited, it rained twice. Then suppose that during six throws of a six-sided die, a single pip came up twice. What would you conclude about the climate? What would you conclude about the die? See, these are different cases.

---

[3]Decimal notation is used here.

Regarding the climate, the best one can do might be to suppose *empirically* that, on average, it rained two days out of every six; whereas one should probably assume *a priori* that, on average, a single pip were to come up one throw out of every six. For the die, one would regard the two-throw observation to represent but a random fluctuation.

Cases of either kind can be quantitatively analyzed. This chapter mostly (though not solely) analyzes cases of the *a priori* kind, cases like that of the die.

Other than by the brief introduction you are reading, this book is not well placed to offer a gentle tutorial in probabilistic thought.[4] What it does offer, in the form of the present chapter, is the discovery and derivation of some of the essential mathematical functions of probability theory, plus a brief investigation of these functions' principal properties and typical use.

20.1 Definitions and basic concepts

A *probability* is the chance that a trial of some kind will result in some specified event of interest. Conceptually,

$$P_{\mathrm{event}} \equiv \lim_{N\to\infty} \frac{N_{\mathrm{event}}}{N},$$

where N and N_{event} are the numbers respectively of trials and events. A *probability density function* (PDF) or *distribution* is a function $f(x)$ defined such that

$$\begin{aligned} P_{ba} &= \int_a^b f(x)\,dx, \\ 1 &= \int_{-\infty}^{\infty} f(x)\,dx, \\ 0 &\le f(x), \quad \Im[f(x)] = 0. \end{aligned} \tag{20.1}$$

where the event of interest is that the *random variable* x fall[5] within the interval[6] $a < x < b$ and P_{ba} is the probability of this event. The corresponding

[4] R. W. Hamming's [70] ably fills such a role.

[5] This sentence and the rest of the section condense somewhat lengthy tracts of an introductory collegiate statistics textbook like [173][2][111][137], among others. If the sentence and section make little sense to you then so likely will the rest of the chapter, but any statistics text you might find conveniently at hand should fill the gap—which is less a mathematical gap than a conceptual one. Or, if defiant, you can stay here and work through the concepts on your own.

[6] We might as well have expressed the interval $a < x < b$ as $a \le x \le b$ or even as $a \le x < b$, except that such notational niceties would distract from the point the notation

cumulative distribution function (CDF) is

$$F(x) \equiv \int_{-\infty}^{x} f(\tau)\,d\tau, \tag{20.2}$$

where

$$\begin{aligned} 0 &= F(-\infty), \\ 1 &= F(\infty), \\ P_{ba} &= F(b) - F(a). \end{aligned} \tag{20.3}$$

The *quantile* $F^{-1}(\cdot)$ inverts the CDF $F(x)$ such that

$$F^{-1}[F(x)] = x, \tag{20.4}$$

generally calculable by a Newton-Raphson iteration (4.30) if by no other means.

It is easy enough to see that the product

$$P = P_1 P_2 \tag{20.5}$$

of two probabilities composes the single probability that not just one but both of two independent events will occur. Harder to see, but just as important, is that the convolution

$$f(x) = f_1(x) * f_2(x) \tag{20.6}$$

of two probability density functions composes the single probability density function of the sum of two random variables

$$x = x_1 + x_2, \tag{20.7}$$

where, per Table 18.3,

$$f_2(x) * f_1(x) = f_1(x) * f_2(x) \equiv \int_{-\infty}^{\infty} f_1\left(\frac{x}{2} - \tau\right) f_2\left(\frac{x}{2} + \tau\right) d\tau.$$

---

means to convey. The notation in this case is not really interested in the bounding points themselves. If *we* are interested in the bounding points, as for example we would be if $f(x) = \delta(x)$ and $a = 0$, then we can always write in the style of $P_{(0^-)b}$, $P_{(0^+)b}$, $P_{(a-\epsilon)(b+\epsilon)}$, $P_{(a+\epsilon)(b-\epsilon)}$ or the like. We can even be most explicit and write in the style of $P\{a \le x \le b\}$, often met in the literature.

That is, if you think about it in a certain way, the probability that $a < x_1 + x_2 < b$ cannot but be

$$
\begin{aligned}
P_{ba} &= \lim_{\epsilon\to 0^+} \sum_{k=-\infty}^{\infty} \left\{ \left[\int_{a-k\epsilon}^{b-k\epsilon} f_1(x)\,dx \right] \left[\int_{(k-1/2)\epsilon}^{(k+1/2)\epsilon} f_2(x)\,dx \right] \right\} \\
&= \lim_{\epsilon\to 0^+} \sum_{k=-\infty}^{\infty} \left\{ \left[\int_{a}^{b} f_1(x-k\epsilon)\,dx \right] \left[\epsilon f_2(k\epsilon) \right] \right\} \\
&= \int_{-\infty}^{\infty} \left[\int_a^b f_1(x-\tau)\,dx \right] f_2(\tau)\,d\tau \\
&= \int_a^b \left[\int_{-\infty}^{\infty} f_1(x-\tau) f_2(\tau)\,d\tau \right] dx \\
&= \int_a^b \left[\int_{-\infty}^{\infty} f_1\left(\frac{x}{2}-\tau\right) f_2\left(\frac{x}{2}+\tau\right) d\tau \right] dx,
\end{aligned}
$$

which in consideration of (20.1) implies (20.6).

20.2 The statistics of a distribution

A probability density function $f(x)$ describes a distribution whose *mean* μ and *standard deviation* σ are defined such that

$$
\begin{aligned}
\mu \equiv \langle x \rangle &= \int_{-\infty}^{\infty} f(x)x\,dx, \\
\sigma^2 \equiv \left\langle (x - \langle x \rangle)^2 \right\rangle &= \int_{-\infty}^{\infty} f(x)(x-\mu)^2\,dx;
\end{aligned}
\tag{20.8}
$$

where $\langle\cdot\rangle$ indicates the *expected value* of the quantity enclosed, defined as the first line of (20.8) suggests. The mean μ is just the distribution's average, about which a random variable should center. The standard deviation σ measures a random variable's typical excursion from the mean. The mean and standard deviation are *statistics* of the distribution.[7] When for example the chapter's introduction proposed that the average 25-year-old U.S. male were 69 ± 3 inches tall, it was saying that his height could quantitatively be modeled as a random variable drawn from a distribution whose statistics are $\mu = 69$ inches and $\sigma = 3$ inches.

The first line of (20.8), defining μ, might seem obvious enough, but one might ask why σ had not instead been defined to be $\langle |x - \langle x \rangle| \rangle$. Would that

[7]Other statistics than the mean and standard deviation are possible, but these two are the most important ones and are the two this book will treat.

not have been more obvious? One answer is[8] that, yes, it might have been more obvious but it would not have been analytic (§§ 2.11.3 and 8.4). Another answer is that one likes to regard long excursions from the mean more seriously than short ones. A third answer is that the second line of (20.8) comports with the elegant mathematics of least squares and Moore-Penrose (§ 13.6). Whatever the answer, (20.8) is the definition conventionally used.

20.3 The sum of random variables

The statistics of the sum of two random variables $x = x_1 + x_2$ are of interest. For the mean, substituting (20.6) into the first line of (20.8),

$$\begin{aligned}
\mu &= \int_{-\infty}^{\infty} [f_1(x) * f_2(x)] x\,dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1\left(\frac{x}{2} - \tau\right) f_2\left(\frac{x}{2} + \tau\right) d\tau\, x\,dx \\
&= \int_{-\infty}^{\infty} f_2(\tau) \int_{-\infty}^{\infty} f_1(x - \tau) x\,dx\,d\tau \\
&= \int_{-\infty}^{\infty} f_2(\tau) \int_{-\infty}^{\infty} f_1(x)(x + \tau)\,dx\,d\tau \\
&= \int_{-\infty}^{\infty} f_2(\tau) \left[\int_{-\infty}^{\infty} f_1(x) x\,dx + \tau \int_{-\infty}^{\infty} f_1(x)\,dx \right] d\tau \\
&= \int_{-\infty}^{\infty} f_2(\tau) [\mu_1 + \tau]\,d\tau \\
&= \mu_1 \int_{-\infty}^{\infty} f_2(\tau)\,d\tau + \int_{-\infty}^{\infty} f_2(\tau)\tau\,d\tau.
\end{aligned}$$

That is,

$$\mu = \mu_1 + \mu_2, \tag{20.9}$$

which is no surprise, but at least it is nice to know that our mathematics is working as it should. The standard deviation of the sum of two random variables is such that, substituting (20.6) into the second line of (20.8),

$$\begin{aligned}
\sigma^2 &= \int_{-\infty}^{\infty} [f_1(x) * f_2(x)] (x - \mu)^2\,dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1\left(\frac{x}{2} - \tau\right) f_2\left(\frac{x}{2} + \tau\right) d\tau\,(x - \mu)^2\,dx.
\end{aligned}$$

[8]The writer does not know the original, historical answer.

Applying (20.9),

$$
\begin{aligned}
\sigma^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1\left(\frac{x}{2}-\tau\right) f_2\left(\frac{x}{2}+\tau\right) d\tau\,(x-\mu_1-\mu_2)^2\,dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1\left(\frac{x+\mu_1}{2}-\tau\right) f_2\left(\frac{x+\mu_1}{2}+\tau\right) d\tau\,(x-\mu_2)^2\,dx \\
&= \int_{-\infty}^{\infty} f_2(\tau)\int_{-\infty}^{\infty} f_1(x+\mu_1-\tau)(x-\mu_2)^2\,dx\,d\tau \\
&= \int_{-\infty}^{\infty} f_2(\tau)\int_{-\infty}^{\infty} f_1(x)[(x-\mu_1)+(\tau-\mu_2)]^2\,dx\,d\tau \\
&= \int_{-\infty}^{\infty} f_2(\tau)\bigg\{\int_{-\infty}^{\infty} f_1(x)(x-\mu_1)^2\,dx \\
&\qquad + 2(\tau-\mu_2)\int_{-\infty}^{\infty} f_1(x)(x-\mu_1)\,dx \\
&\qquad + (\tau-\mu_2)^2\int_{-\infty}^{\infty} f_1(x)\,dx\bigg\}d\tau \\
&= \int_{-\infty}^{\infty} f_2(\tau)\bigg\{\int_{-\infty}^{\infty} f_1(x)(x-\mu_1)^2\,dx \\
&\qquad + 2(\tau-\mu_2)\int_{-\infty}^{\infty} f_1(x)x\,dx \\
&\qquad + (\tau-\mu_2)(\tau-\mu_2-2\mu_1)\int_{-\infty}^{\infty} f_1(x)\,dx\bigg\}d\tau.
\end{aligned}
$$

Applying (20.8) and (20.1),

$$
\begin{aligned}
\sigma^2 &= \int_{-\infty}^{\infty} f_2(\tau)\left\{\sigma_1^2+2(\tau-\mu_2)\mu_1+(\tau-\mu_2)(\tau-\mu_2-2\mu_1)\right\}d\tau \\
&= \int_{-\infty}^{\infty} f_2(\tau)\left\{\sigma_1^2+(\tau-\mu_2)^2\right\}d\tau \\
&= \sigma_1^2\int_{-\infty}^{\infty} f_2(\tau)\,d\tau+\int_{-\infty}^{\infty} f_2(\tau)(\tau-\mu_2)^2\,d\tau.
\end{aligned}
$$

Applying (20.8) and (20.1) again,

$$\sigma^2 = \sigma_1^2+\sigma_2^2. \tag{20.10}$$

If this is right—as indeed it is—then the act of adding random variables together not only adds the means of the variables' respective distributions according to (20.9) but also, according to (20.10), adds the squares of the standard deviations. It follows inductively that, if N independent instances $x_1, x_2, \ldots, x_N$ of a random variable are drawn from the same distribution $f_o(x_k)$, the distribution's statistics being μ_o and σ_o, then the statistics of their sum $x = \sum_{k=1}^{N} x_k = x_1 + x_2 + \cdots + x_N$ are

$$\begin{aligned} \mu &= N\mu_o, \\ \sigma &= \left(\sqrt{N}\right)\sigma_o. \end{aligned} \tag{20.11}$$

20.4 The transformation of a random variable

If x_o is a random variable obeying the distribution $f_o(x_o)$ and $g(\cdot)$ is some invertible function whose inverse per (2.63) is styled $g^{-1}(\cdot)$, then

$$x \equiv g(x_o)$$

is itself a random variable obeying the distribution

$$f(x) = \left.\frac{f_o(x_o)}{|dg/dx_o|}\right|_{x_o=g^{-1}(x)}. \tag{20.12}$$

Another, suaver way to write the same thing is as that

$$f(x)\,|dx| = f_o(x_o)\,|dx_o|\,. \tag{20.13}$$

Either way, this is almost obvious if seen from just the right perspective, but can in any case be supported symbolically by

$$\int_a^b f_o(x_o)\,dx_o = \left|\int_{g(a)}^{g(b)} f_o(x_o)\frac{dx_o}{dx}\,dx\right| = \int_{g(a)}^{g(b)} f_o(x_o)\left|\frac{dx_o}{dg}\right|\,dx$$

since, on the other hand

$$\int_a^b f_o(x_o)\,dx_o = \int_{g(a)}^{g(b)} f(x)\,dx.$$

One of the most frequently useful transformations is the simple

$$x \equiv g(x_o) \equiv \alpha x_o, \quad \Im(\alpha) = 0. \tag{20.14}$$

For this, evidently $dg/dx_o = \alpha$ or $dx = \alpha\,dx_o$, so according to (20.12) or (20.13)

$$f(x) = \frac{1}{|\alpha|}f_o\left(\frac{x}{\alpha}\right). \tag{20.15}$$

If $\mu_o = 0$ and $\sigma_o = 1$, then $\mu = 0$ and, applying (20.13) in train of (20.8),

$$\sigma^2 = \int_{-\infty}^{\infty} f(x)x^2\,dx = \int_{-\infty}^{\infty} f_o(x_o)(\alpha x_o)^2\,dx_o = \alpha^2;$$

whereby $\sigma = |\alpha|$ and, if $\alpha > 0$, one can rewrite the transformed PDF as

$$f(x) = \frac{1}{\sigma} f_o\left(\frac{x}{\sigma}\right) \text{ and } \mu = 0, \text{ if } \mu_o = 0 \text{ and } \sigma_o = 1. \tag{20.16}$$

Assuming null mean, (20.16) states that the act of scaling a random variable flattens out the variable's distribution and scales its standard deviation, all by the same factor—which, naturally, is what one would expect such an act to do.

20.5 The normal distribution

Combining the ideas of §§ 20.3 and 20.4 can lead one to ask whether a zero-mean distribution does not exist for which, when independent random variables drawn from it are added together, *the sum obeys the same distribution,* only the standard deviations differing. More precisely, the ideas can lead one to seek a distribution

$$f_o(x_o)\colon\ \mu_o = 0,\ \sigma_o = 1;$$

for which, if x_1 and x_2 are random variables drawn respectively from the distributions

$$\begin{aligned} f_1(x_1) &= \frac{1}{\sigma_1} f_o\left(\frac{x_1}{\sigma_1}\right), \\ f_2(x_2) &= \frac{1}{\sigma_2} f_o\left(\frac{x_2}{\sigma_2}\right), \end{aligned}$$

as (20.16) suggests, then

$$x = x_1 + x_2$$

is by construction a random variable drawn from the distribution

$$f(x) = \frac{1}{\sigma} f_o\left(\frac{x}{\sigma}\right),$$

where per (20.10),

$$\sigma^2 = \sigma_1^2 + \sigma_2^2.$$

There are several distributions one might try, but eventually the Gaussian pulse $\Omega(x_o)$ of §§ 17.3 and 18.4,

$$\Omega(x) = \frac{\exp\left(-x^2/2\right)}{\sqrt{2\pi}}, \tag{20.17}$$

recommends itself. This works. The distribution $f_o(x_o) = \Omega(x_o)$ meets our criterion.

20.5.1 Proof

To prove that the distribution $f_o(x_o) = \Omega(x_o)$ meets our criterion we shall have first to show that it is indeed a distribution according to (20.1). Especially, we shall have to demonstrate that

$$\int_{-\infty}^{\infty} \Omega(x_o)\, dx_o = 1.$$

Fortunately as it happens we have already demonstrated this fact in an earlier chapter, while working on Fourier transforms, as (18.59). The function $\Omega(x_o)$ had interested us during the earlier chapter because it is an analytic function that autotransforms, so now in this chapter we observe that, since $\Omega(x_o)$ evidently meets the other demands of (20.1), $\Omega(x_o)$ is apparently indeed also a proper distribution, whatever its other properties might be. That $\mu_o = 0$ for $\Omega(x_o)$ is obvious by symmetry. That $\sigma_o = 1$ is shown by

$$\begin{aligned}
\sigma^2 \equiv \int_{-\infty}^{\infty} & \Omega(x_o) x_o^2\, dx_o \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x_o^2}{2}\right) x_o^2\, dx_o \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \left[-x_o\right] \left[-x_o \exp\left(-\frac{x_o^2}{2}\right) dx_o\right] \right\} \\
&= -\frac{1}{\sqrt{2\pi}} \int_{x_o=-\infty}^{\infty} x_o\, d\left[\exp\left(-\frac{x_o^2}{2}\right)\right] \\
&= -\frac{x_o \exp\left(-x_o^2/2\right)}{\sqrt{2\pi}}\bigg|_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x_o^2}{2}\right) dx_o \\
&= 0 + \int_{-\infty}^{\infty} \Omega(x_o)\, dx_o,
\end{aligned}$$

the integration via the by-parts method of § 9.4, the result according to (18.59) that

$$\sigma^2 \equiv \int_{-\infty}^{\infty} \Omega(x_o) x_o^2 \, dx_o = 1, \tag{20.18}$$

implying that $\sigma = 1$ as was to be shown. Now having justified the assertions that $\Omega(x_o)$ is a proper distribution and that its statistics are $\mu_o = 0$ and $\sigma_o = 1$, all that remains to be proved per (20.6) is that

$$\begin{gathered}\left[\frac{1}{\sigma_1}\Omega\left(\frac{x_o}{\sigma_1}\right)\right] * \left[\frac{1}{\sigma_2}\Omega\left(\frac{x_o}{\sigma_2}\right)\right] = \frac{1}{\sigma}\Omega\left(\frac{x_o}{\sigma}\right), \\ \sigma_1^2 + \sigma_2^2 = \sigma^2,\end{gathered} \tag{20.19}$$

which is to prove that the sum of Gaussian random variables is itself Gaussian. We will prove it in the Fourier domain of chapter 18 as follows. According to Tables 18.1, 18.3 and 18.5, and to (20.17),

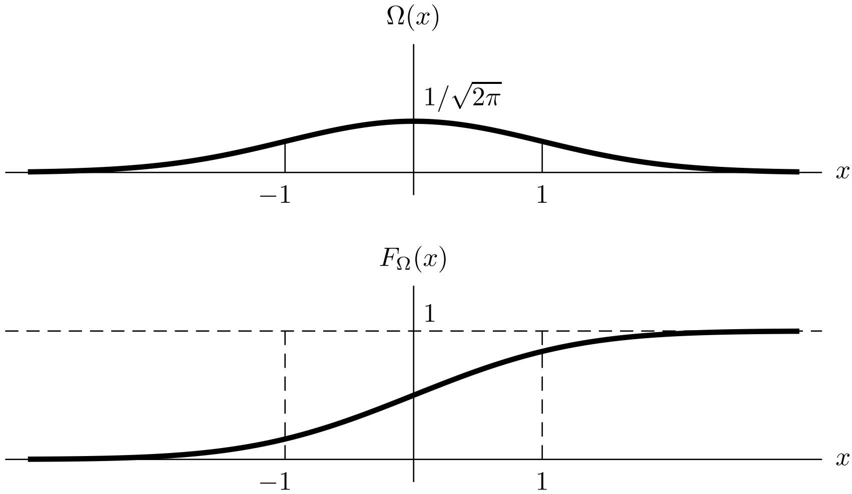
$$\begin{aligned}
&\left[\frac{1}{\sigma_1}\Omega\left(\frac{x_o}{\sigma_1}\right)\right] * \left[\frac{1}{\sigma_2}\Omega\left(\frac{x_o}{\sigma_2}\right)\right] \\
&\quad= \mathscr{F}^{-1}\left\{\left(\sqrt{2\pi}\right)\mathscr{F}\left[\frac{1}{\sigma_1}\Omega\left(\frac{x_o}{\sigma_1}\right)\right]\mathscr{F}\left[\frac{1}{\sigma_2}\Omega\left(\frac{x_o}{\sigma_2}\right)\right]\right\} \\
&\quad= \mathscr{F}^{-1}\left\{\left(\sqrt{2\pi}\right)\Omega(\sigma_1 x_o)\Omega(\sigma_2 x_o)\right\} \\
&\quad= \mathscr{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{\sigma_1^2 x_o^2}{2}\right]\exp\left[-\frac{\sigma_2^2 x_o^2}{2}\right]\right\} \\
&\quad= \mathscr{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{\left(\sigma_1^2+\sigma_2^2\right)x_o^2}{2}\right]\right\} \\
&\quad= \mathscr{F}^{-1}\left\{\Omega\left[\left(\sqrt{\sigma_1^2+\sigma_2^2}\right)x_o\right]\right\} \\
&\quad= \frac{1}{\sqrt{\sigma_1^2+\sigma_2^2}}\,\Omega\left(\frac{x_o}{\sqrt{\sigma_1^2+\sigma_2^2}}\right),
\end{aligned}$$

the last line of which is (20.19) in other notation, thus completing the proof.

20.5.2 Plots and remarks

In the Fourier context of chapter 18 one usually names $\Omega(\cdot)$ the *Gaussian pulse,* as we have seen. The function $\Omega(\cdot)$ turns out to be even more prominent in probability theory than in Fourier theory, however, and in a probabilistic context it usually goes by the name of the *normal distribution.* This

Figure 20.1: The normal distribution $\Omega(x) \equiv (1/\sqrt{2\pi})\exp(-x^2/2)$ and its cumulative distribution function $F_\Omega(x) = \int_{-\infty}^{x} \Omega(\tau)\,d\tau$.



is what we will call $\Omega(\cdot)$ through the rest of the present chapter. Alternate conventional names include those of the *Gaussian distribution* and the *bell curve* (the Greek capital Ω vaguely, accidentally resembles a bell, as does the distribution's plot, and we will not be too proud to take advantage of the accident, so that is how you can remember it if you like). By whichever name, Fig. 20.1 plots the normal distribution $\Omega(\cdot)$ and its cumulative distribution function (20.2).

Regarding the cumulative normal distribution function, one way to calculate it numerically is to integrate the normal distribution's Taylor series term by term. As it happens, § 9.12 has worked a similar integral as an example, so this section will not repeat the details, but the result is that

$$\begin{aligned} F_\Omega(x_o) = \int_{-\infty}^{x_o} \Omega(\tau)\,d\tau &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-)^k x_o^{2k+1}}{(2k+1)2^k k!} \\ &= \frac{1}{2} + \frac{x_o}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{1}{2k+1} \prod_{j=1}^{k} \frac{-x_o^2}{2j}. \end{aligned} \tag{20.20}$$

Unfortunately, this Taylor series—though always theoretically correct—is practical only for small and moderate $|x_o| \lesssim 1$. For $|x_o| \gg 1$, see § 20.10.

The normal distribution tends to be the default distribution in applied mathematics. When one lacks a reason to do otherwise, one models a random quantity as a normally distributed random variable. Section 20.7 tells more.

20.5.3 Motive

Equation (20.17) seems almost unfair to posit. Once the equation has been posited, the proof follows, the proof validating the position; so the logic is valid, but why posit the equation in the first place?[9]

One answer is that the equation (20.17) is not really all that obscure. To study expressions that resemble $\exp(-x^2/2)$ for their own sakes is neither unreasonable nor especially unlikely. Some mathematician or other must probably, eventually have thought to try such an expression against the logic of § 20.5.1. One he had tried it and had shown us his result, we would know to posit it.

The last paragraph's answer is actually a pretty good answer. We should not be embarrassed to give it. Much of mathematics goes that way, after all.

Nevertheless, an alternate answer is known. Suppose that N coins are tossed and that $2m$ is the number of heads in excess of the number of tails (for example, if 6 heads and 2 tails, then $2m = 6-2 = 4$ and $N = 6+2 = 8$). According to the combinatorics of § 4.2,

$$f(m) = \binom{N}{[N+2m]/2} = \frac{N!/[(N-2m)/2]!}{[(N+2m)/2]!}$$

computes the probability that m will have a given value.

[9] This subsection is optional reading for the benefit of the curious. You can skip it without burdening the rest of the book.

Since we are merely motivating, we need not be precise, so approximately,

$$\begin{aligned}
\frac{d}{dm}\ln f(m) &= \frac{df/dm}{f(m)} \\
&\approx \frac{f(m+1)-f(m-1)}{2f(m)} \\
&\approx \frac{1}{2}\left[\frac{f(m+1)}{f(m)} - \frac{f(m-1)}{f(m)}\right] \\
&\approx \frac{1}{2}\left[\frac{(N-2m)/2}{(N+2m+2)/2} - \frac{(N+2m)/2}{(N-2m+2)/2}\right] \\
&\approx \frac{1}{2}\left[\frac{1-2m/N}{1+(2m+2)/N} - \frac{1+2m/N}{1-(2m+2)/N}\right] \\
&\approx \frac{1}{2}\left[\frac{(-8m-4)/N}{1-[(2m+2)/N]^2}\right] \approx \frac{1}{2}\left[\frac{-8m-4}{N}\right]\left[1+\left(\frac{2m+2}{N}\right)^2\right] \\
&\approx \frac{1}{2}\left[\frac{-8m}{N}\right] = -\frac{4m}{N}.
\end{aligned}$$

Changing $x \leftarrow m$ and $\alpha \leftarrow 4/N$,

$$\frac{d}{dx}\ln f(x) = \frac{df/dx}{f(x)} \approx -\alpha x.$$

A function that does this is

$$f(x) \approx C\exp(-\alpha x^2/2),$$

which motivates (20.17).

20.6 Inference of statistics

Suppose that several, concrete instances of a random variable—the instances collectively called a *sample*—were drawn from a distribution $f(x)$ and presented to you, but that you were not told the shape of $f(x)$. Could you infer the shape?

The answer is that you could infer the shape with passable accuracy provided that the number N of instances were large. Typically however one will be prepared to make some assumption about the shape such as that

$$f(x) = \mu + \frac{1}{\sigma}\Omega\left(\frac{x}{\sigma}\right), \tag{20.21}$$

which is to assume that x were normally distributed with unknown statistics μ and σ. The problem then becomes to infer the statistics from the sample.

20.6.1 Inference of the mean

In the absence of additional information, one can hardly suppose much regarding the mean other than that

$$\mu \approx \frac{1}{N}\sum_k x_k. \tag{20.22}$$

One infers the mean to be the average of the instances observed.

20.6.2 An imputed ensemble

One might naïvely think to infer a standard deviation in much the same way as § 20.6.1 has inferred a mean, except that to calculate the standard deviation directly according to (20.8) would implicate our imperfect estimate (20.22) of the mean. If we wish to estimate the standard deviation accurately from the sample then we shall have to proceed more carefully than that.

Section 20.6.3 will estimate the standard deviation after the subsection you are reading has prepared the ground on which to do it. To prepare the ground, let us now define the shifted random variable

$$u \equiv x - \mu_{\mathrm{true}}$$

in lieu of the random variable x, where μ_{true} *is not the estimated mean* of (20.22) but is the true, unknown mean of the hidden distribution $f(x)$—such that an instance u_k of the random variable u is in no way independent of, but is rather wholly dependent on, the corresponding instance x_k of the random variable x; but also, paradoxically, such that the exact value of u_k remains unknown even if the exact value of x_k is known. And why does the exact value of u_k remain unknown? It remains unknown because the separation μ_{true} (which from the present perspective is no random variable but a fixed number) between u_k and x_k remains unknown. At any rate, the distribution of u is

$$f_u(u) \equiv f(u + \mu_{\mathrm{true}}),$$

a distribution which by construction *is known to have zero true mean,*

$$\langle u \rangle = 0,$$

even though the standard deviation σ_{true} the two distributions $f(x)$ and $f_u(u)$ share remains unknown.

Statistical reasoning is tricky, isn't it? No? Quite straightforward, you say? Good, let us continue.

Regarding not any particular sample of N instances but a conceptually infinite *ensemble* of samples, each sample consisting of N instances, two identities the standard-deviational analysis of § 20.6.3 will be able to use are that

$$\left\langle \sum_k u_k^2 \right\rangle = N\sigma_{\text{true}}^2,$$

$$\left\langle {\sum_k}^2 u_k \right\rangle = N\sigma_{\text{true}}^2,$$

where σ_{true} is (as the notation suggests and as an earlier paragraph has observed) the true, unknown standard deviation of the hidden distribution $f(x)$ and thus also of the shifted distribution $f_u(u)$. The first of the two identities is merely a statement of the leftward part of (20.8)'s second line with respect to the distribution $f_u(u)$ whose mean $\langle u \rangle = 0$ is, as we said, known to be zero despite that the distribution itself remains unknown. The second of the two identities considers the sum $\sum_k u_k$ itself as a random variable whose mean again is zero but whose standard deviation σ_Σ according to (20.10) is such that $\sigma_\Sigma^2 = N\sigma_{\text{true}}^2$.

Admittedly, one might wonder how we can speak sensibly of an ensemble when no concrete ensemble is to be observed. Observation, after all, sees only the one sample of N instances. However, we have assumed that a hidden distribution $f(x)$ exists and that the several instances x_k, which *are* observed, have been drawn from it. Our assumption might be *wrong,* of course—in general this is very difficult to judge—but we have assumed it and the assumption has consequences. Among the consequences is that $f(x)$ possesses statistics μ_{true} and σ_{true}. We do not know—we shall never know—the right values of these statistics; but our assumption implies that they do exist and do have values, values one can and should write symbols to represent.

Section 20.6.3 will employ the symbols, next.

20.6.3 Inference of the standard deviation

With the definitions and identities of § 20.6.2 in hand, let us construct from the available sample the quantity

$$(\sigma')^2 \equiv \frac{1}{N}\sum_k \left(x_k - \frac{1}{N}\sum_\ell x_\ell \right)^2,$$

modeled on (20.8). Evidently,

$$\lim_{N\to\infty} \sigma' = \sigma_{\text{true}}.$$

However, unlike the σ_{true}, the σ' is a quantity we can actually compute from an observed sample. Let the sample consist of $N > 1$ instances. By successive steps,

$$\begin{aligned}
(\sigma')^2 &= \frac{1}{N}\sum_k \left([u_k + \mu_{\text{true}}] - \frac{1}{N}\sum_\ell [u_\ell + \mu_{\text{true}}] \right)^2 \\
&= \frac{1}{N}\sum_k \left(u_k - \frac{1}{N}\sum_\ell u_\ell \right)^2 \\
&= \frac{1}{N}\sum_k \left(u_k^2 - \frac{2}{N}u_k \sum_\ell u_\ell + \frac{1}{N^2}\sum_\ell{}^2 u_\ell \right) \\
&= \frac{1}{N}\sum_k u_k^2 - \frac{2}{N^2}\sum_k u_k \sum_\ell u_\ell + \frac{1}{N^2}\sum_\ell{}^2 u_\ell \\
&= \frac{1}{N}\sum_k u_k^2 - \frac{2}{N^2}\sum_k{}^2 u_k + \frac{1}{N^2}\sum_k{}^2 u_k \\
&= \frac{1}{N}\sum_k u_k^2 - \frac{1}{N^2}\sum_k{}^2 u_k,
\end{aligned}$$

the expected value of which over a conceptually infinite ensemble of samples (each sample consisting—as explained by § 20.6.2—of an equal number $N >$ 1 of instances) is

$$\left\langle (\sigma')^2 \right\rangle = \frac{1}{N}\left\langle \sum_k u_k^2 \right\rangle - \frac{1}{N^2}\left\langle \sum_k{}^2 u_k \right\rangle.$$

Applying the identities of § 20.6.2,

$$\left\langle (\sigma')^2 \right\rangle = \sigma_{\text{true}}^2 - \frac{\sigma_{\text{true}}^2}{N} = \frac{N-1}{N}\sigma_{\text{true}}^2,$$

from which

$$\sigma_{\text{true}}^2 = \frac{N}{N-1} \left\langle (\sigma')^2 \right\rangle.$$

Because the expectation $\langle(\sigma')\rangle$ is not a quantity whose value we know, we can only suppose that $\langle(\sigma')^2\rangle \approx (\sigma')^2$, whereby

$$\sigma_{\text{true}}^2 \approx \frac{N}{N-1} (\sigma')^2.$$

Substituting the definition of $(\sigma')^2$ into the last equation and changing symbols $\sigma \leftarrow \sigma_{\text{true}}$, we have that

$$\sigma^2 \approx \frac{1}{N-1} \sum_k \left(x_k - \frac{1}{N} \sum_\ell x_\ell \right)^2. \tag{20.23}$$

This σ^2 is apparently a little greater (though, provided that N is sufficiently large, not much greater) than a naïve assumption that σ equaled σ' would have supposed.

Notice according (20.23) that σ, unlike σ', infers a standard deviation only when the sample includes at least two instances! Indeed, σ is sensible to do so, for the one case in which a naïve analysis were right would be when the true mean μ_{true} were for some reason *a priori* exactly known, leaving only the standard deviation to be inferred. In such a case,

$$\sigma^2 \approx \frac{1}{N} \sum_k (x_k - \mu_{\text{true}})^2. \tag{20.24}$$

The estimates (20.22) and (20.23) are known as *sample statistics.* They are the statistics one imputes to an unknown distribution based on the incomplete information a sample of $N > 1$ instances affords.

20.6.4 Correlation and its inference

The chapter you are reading has made some assumptions, not all of which it has explicitly stated, or at any rate not all of which it has fully developed. One assumption the chapter has made is that instances of its random variables have been *independent.* In statistical work however one must sometimes handle *correlated* quantities like the height and weight of a 25-year-old U.S. male—for, obviously, if I point to some 25-year-old over there and say, "That's Pfufnik. The average is 187 pounds, but he weighs 250!" then your estimate of his probable height will change, because height and weight are not independent but correlated. The conventional statistical measure[10] of the correlation of a sample of N pairs (x_k, y_k) of data, such as the $([\text{height}]_k, [\text{weight}]_k)$ of the example, is the *correlation coefficient*

$$r \equiv \frac{\sum_k (x_k - \mu_x)(y_k - \mu_y)}{\sqrt{\sum_k (x_k - \mu_x)^2 \sum_k (y_k - \mu_y)^2}}, \tag{20.25}$$

a unitless quantity whose value is ± 1—the ± 1 indicating perfect correlation—when $y_k = x_k$ or even $y_k = a_1 x_k + a_0$; but whose value should be near zero when the paired data are unrelated. See Fig. 13.1 for another example of the kind of paired data in whose correlation one might be interested: in the figure, the correlation would be $+1$ if the points fell all right on the line. (Beware that the conventional correlation coefficient of eqn. 20.25 can overstate the relationship between paired data when N is small. Consider for instance that $r = \pm 1$ always when $N = 2$. The coefficient as given is nevertheless conventional.)

That $r = \pm 1$ when $y_k = a_1 x_k + a_0$ is seen by observing that

$$\begin{aligned} y_k - \mu_y &= (a_1)\left[x_k - \frac{\mu_y - a_0}{a_1}\right] = (a_1)\left[x_k - \frac{\left(\sum_\ell y_\ell\right)/N - a_0}{a_1}\right] \\ &= (a_1)\left[x_k - \frac{\left(\sum_\ell a_1 x_\ell\right)/N}{a_1}\right] = (a_1)\left[x_k - \mu_x\right], \end{aligned}$$

which, when substituted into (20.25), yields the stipulated result.

20.6.5 Remarks

If further elaborated, the mathematics of statistics rapidly grows much more complicated. The book will not pursue the matter further but will mention that the kinds of questions that arise can, among others, involve the statistics of the statistics themselves, treating the *statistics* as random variables. Section 20.6.3 has done this just a bit. The book will avoid doing more of it.

Such questions confound two, separate uncertainties: the uncertainty inherent by definition (20.1) in a random variable even were the variable's distribution precisely known; and the uncertain knowledge of the distribution.[11] Fortunately, if $N \gg 1$, then one can usually tear the two uncertainties from one another without undue violence to accuracy, pretending that

[10] [173, § 9.9][2, eqns. 12-6 and 12-14]

[11] The subtle mathematical implications of this far exceed the scope of the present book but are developed to one degree or another in numerous collegiate statistics texts of which [31][173][2][111][137] are representative examples.

one knew the unknown statistics μ and σ to have exactly the values (20.22) and (20.23) respectively calculate for them, supposing that the distribution were the normal (20.21), and modeling on this basis.

Unfortunately, that $N \gg 1$ is not so for many samples of practical interest. As the biologist M. G. Bulmer recounts,

> 'Student' had found, however, that in his practical work for Guinness' brewery he was often forced to deal with samples far too small for the customary large sample approximations to be applicable. It was gradually realised after the publication of his paper, and of R. A. Fisher's papers on other problems in small sample theory, that if the sample were large enough the answer to any question one might ask would be obvious, and that it was only in the case of small and moderate-sized samples that any statistical problem arose. [31, chapter 9]

Notwithstanding, the book you are reading will delve no further into the matter, turning attention rather back to probabilistic topics proper. It is worth noting before we turn, however, that it took a pair of biologists—'Student' and Fisher—to broaden the relevant mathematics. Whatever logical service the mathematics profession may subsequently have rendered, whatever formal buttresses it may subsequently have built, professional mathematicians on their own might neither have discerned the direction in which to explore nor have discovered the ground upon which to build. See §§ 22.3 and 22.5.

20.7 The random walk and its consequences

This section analyzes the simple but oft-encountered statistics of a series of all-or-nothing attempts.

20.7.1 The random walk

Matthew Sands gave a famous lecture [57, § I:6] to freshmen in physics, on probability, on behalf of Richard P. Feynman at Caltech in the fall of 1961. The lecture is a classic and is recommended to every reader who can conveniently lay hands on a copy—recommended among other reasons because the lecture lends needed context to the rather abstruse mathematics this chapter has presented to the present point. One section of the lecture begins, "There is [an] interesting problem in which the idea of probability is required. It is the problem of the 'random walk.' In its simplest version, we imagine a 'game' in which a 'player' starts at the point $[D = 0]$ and at each 'move' is required to take a step *either* forward (toward $[+D]$) *or* backward (toward $[-D]$). The choice is to be made *randomly,* determined, for example, by the toss of a coin. How shall we describe the resulting motion?"

Sands goes on to observe that, though one cannot guess whether the "player" will have gone forward or backward after N steps—and, indeed, that in the absence of other information one must expect that $\langle D_N \rangle = 0$, indicating zero expected net progress—"[one has] the feeling that as N increases, [the 'player'] is likely to have strayed farther from the starting point." Sands is right, but if $\langle D_N \rangle$ is not a suitable measure of this "likely stray," so to speak, then what would be?

The measure $\langle |D_N| \rangle$ might recommend itself but this, being nonanalytic (§§ 2.11.3 and 8.4), proves inconvenient in practice (you can try it if you like). Fortunately, an alternate, analytic measure, $\langle D_N^2 \rangle$, presents itself. The success of the least-squares technique of § 13.6 encourages us to try it. When tried, the alternate, analytic measure prospers.

Section 20.2 has actually already introduced $\langle D_N^2 \rangle$ in another guise as σ^2 (the σ^2 in this context being the standard deviation of an ensemble of a conceptually infinite number of instances of D_N, each instance being the sum of N random steps). The squared distance D_N^2 is nonnegative, a quality necessary to a good index of stray. The squared distance D_N^2 and its expectance $\langle D_N^2 \rangle$ are easy to calculate and, comparatively, also convenient to use. Moreover, scientists and engineers have long been used to accepting such quantities and equivalents like σ^2 as statistical characterizations. We will use them for these reasons among others.

In his lecture, Sands observes that, if the symbol D_{N-1} represents the "player's" position after $N-1$ steps, if next step is ± 1 in size, then the "player's" position after N steps must be $D_N = D_{N-1} \pm 1$. The expected value $\langle D_N \rangle = 0$ is uninteresting as we said, but the expected value $\langle D_N^2 \rangle$ is interesting. And what is this expected value? Sands finds two possibilities: either the "player" steps forward on his Nth step, in which case

$$\left\langle D_N^2 \right\rangle = \left\langle (D_{N-1} + 1)^2 \right\rangle = \left\langle D_{N-1}^2 \right\rangle + 2\left\langle D_{N-1} \right\rangle + 1;$$

or he steps backward on his Nth step, in which case

$$\left\langle D_N^2 \right\rangle = \left\langle (D_{N-1} - 1)^2 \right\rangle = \left\langle D_{N-1}^2 \right\rangle - 2\left\langle D_{N-1} \right\rangle + 1.$$

Since forward and backward are equally likely, the actual expected value must be the average

$$\left\langle D_N^2 \right\rangle = \left\langle D_{N-1}^2 \right\rangle + 1$$

of the two possibilities. Evidently, the expected value increases by 1 with each step. Thus by induction, since $\langle D_0^2 \rangle = 0$,

$$\left\langle D_N^2 \right\rangle = N.$$

Observe that the PDF of a single step x_k is $f_o(x_o) = [\delta(x_o+1)+\delta(x_o-1)]/2$, where $\delta(\cdot)$ is the Dirac delta of Fig. 7.11; and that the corresponding statistics are $\mu_o = 0$ and $\sigma_o = 1$. The PDF of D_N is more complicated (though not especially hard to calculate in view of § 4.2), but its statistics are evidently $\mu_N = 0$ and $\sigma_N = \sqrt{N}$, agreeing with (20.11).

20.7.2 Consequences

An important variation of the random walk comes with the distribution

$$f_o(x_o) = (1-p_o)\delta(x_o) + p_o\delta(x_o - 1), \tag{20.26}$$

which describes or governs an act whose probability of success is p_o. This distribution's statistics according to (20.8) are such that

$$\begin{aligned} \mu_o &= p_o, \\ \sigma_o^2 &= (1-p_o)p_o. \end{aligned} \tag{20.27}$$

As an example of the use,[12] consider a real-estate agent who expects to sell one house per 10 times he shows a house to a prospective buyer: $p_o = 1/10 = 0.10$. The agent's expected result from a single showing, according to (20.27), is to sell $\mu_o \pm \sigma_o = 0.10 \pm 0.30$ of a house. The agent's expected result from $N = 400$ showings, according to (20.11), is to sell $\mu \pm \sigma = N\mu_o \pm \left(\sqrt{N}\right)\sigma_o = 40.0 \pm 6.0$ houses. Such a conclusion, of course, is valid only to the extent to which the model is valid—which in a real-estate agent's case might be *not very*—but that nevertheless is how the mathematics of it work.

As the number N of attempts grows large one finds that the distribution $f(x)$ of the number of successes begins more and more to take on the bell-shape of Fig. 20.1's normal distribution. Indeed, this makes sense, for one would expect the aforementioned real-estate agent to enjoy a relatively high probability of selling 39, 40 or 41 houses but a low probability to sell 10 or 70. Of course, not all distributions that make 39, 40 or 41 more likely than 10 or 70 are normal; but the logic of § 20.5 does suggest that, if there

[12]Decimal notation is again here employed. Maybe, abstractly, $2\pi \approx$ 0x6.487F, but even a hexadecimal enthusiast is unlikely to numeralize real-estate sales in hexadecimal.

were a shape toward which such a distribution tended as N increased, then that shape could hardly be other than the shape of the normal distribution. We will leave the argument in that form.[13]

For such reasons, applications tend to approximate sums of several random variables as though the sums were normally distributed; and, moreover, tend to impute normal distributions to random variables whose true distributions are unnoticed, uninteresting or unknown. In the theory and application of probability, the normal distribution is the master distribution, the distribution of last resort, often the only distribution tried. The banal suggestion, "When unsure, go normal!" usually prospers in probabilistic work.

20.8 Other distributions

Many distributions other than the normal one of Fig. 20.1 are possible. This section will name a few of the most prominent.

20.8.1 The uniform distribution

The *uniform distribution* can be defined in any of several forms, but the conventional form is

$$f(x) = \Pi\left(x - \frac{1}{2}\right) = \begin{cases} 1 & \text{if } 0 \le x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{20.28}$$

where $\Pi(\cdot)$ is the square pulse of Fig. 17.3. Besides sometimes being useful in its own right, this is also the distribution a computer's pseudorandom-number generator obeys. One can extract normally distributed (§ 20.5) or Rayleigh-distributed (§ 20.8.4) random variables from it by the Box-Muller transformation of § 20.9.

[13]Admittedly, the argument, which supposes that all (or at least most) aggregate PDFs must tend toward some common shape as N grows large, is somewhat specious, or at least unrigorous—though on the other hand it is hard to imagine any plausible conclusion other than the correct one the argument reaches—but one might construct an alternate though tedious argument toward the normal distribution on the pattern of § 20.5.3 or on another pattern. To fill in the tedious details is left as an exercise to the interested (penitent?) reader. The author confesses that he prefers the specious argument of the narrative.

20.8.2 The exponential distribution

The *exponential distribution* is

$$f(x) = \frac{u(t)}{\mu} \exp\left(-\frac{x}{\mu}\right), \tag{20.29}$$

the $u(t)$ being Heaviside's unit step (7.21). The distribution's mean is

$$\frac{1}{\mu}\int_0^\infty \exp\left(-\frac{x}{\mu}\right) x\,dx = -\exp\left(-\frac{x}{\mu}\right)(x+\mu)\bigg|_0^\infty = \mu$$

as advertised and its standard deviation is such that

$$\begin{aligned}\sigma^2 &= \frac{1}{\mu}\int_0^\infty \exp\left(-\frac{x}{\mu}\right)(x-\mu)^2\,dx \\ &= -\exp\left(-\frac{x}{\mu}\right)(x^2+\mu^2)\bigg|_0^\infty\end{aligned}$$

(the integration by the method of unknown coefficients of § 9.5 or, quicker, by Table 9.1), which implies that

$$\sigma = \mu. \tag{20.30}$$

The exponential's CDF (20.2) and quantile (20.4) are evidently

$$\begin{aligned}F(x) &= 1 - \exp\left(-\frac{x}{\mu}\right), \quad x \ge 0; \\ F^{-1}(v) &= -\mu\ln(1-v).\end{aligned} \tag{20.31}$$

Among other effects, the exponential distribution models the delay until some imminent event like a mechanical bearing's failure or the arrival of a retail establishment's next customer.

20.8.3 The Poisson distribution

The Poisson distribution is[14]

$$f(x) = \exp(-\mu)\sum_{k=0}^{\infty}\frac{\mu^x\delta(x-k)}{x!}. \tag{20.32}$$

It comes from the consideration of a large number $N \gg 1$ of individually unlikely trials, each trial having a probability $0 < \epsilon \ll 1$ of success, such that the expected number of successes is $\mu = \epsilon N$.

[14] [31, chapter 6]

- The chance that no trial will succeed is evidently

$$\lim_{\eta\to 0^+}\int_{-\eta}^{\eta} f(x)\,dx = (1-\epsilon)^N \approx \exp(-\epsilon N) = \exp(-\mu).$$

- The chance that exactly one trial will succeed is

$$\begin{aligned}\lim_{\eta\to 0^+}\int_{1-\eta}^{1+\eta} f(x)\,dx &= \binom{N}{1}(\epsilon)(1-\epsilon)^{N-1}\\ &\approx \epsilon N\exp(-\epsilon N) = \mu\exp(-\mu).\end{aligned}$$

- The chance that exactly two trials will succeed is

$$\begin{aligned}\lim_{\eta\to 0^+}\int_{2-\eta}^{2+\eta} f(x)\,dx &= \binom{N}{2}(\epsilon^2)(1-\epsilon)^{N-2}\\ &\approx \frac{(\epsilon N)^2}{2!}\exp(-\epsilon N) = \frac{\mu^2\exp(-\mu)}{2!}.\end{aligned}$$

- And so on.

In the limit as $N \to \infty$ and $\epsilon \to 0^+$, the product $\mu = \epsilon N$ remaining finite, the approximations become exact and (20.32) results.

Integrating (20.32) to check,

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= \exp(-\mu)\int_{-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{\mu^x\delta(x-k)}{x!}\,dx\\ &= \exp(-\mu)\sum_{k=0}^{\infty}\int_{-\infty}^{\infty}\frac{\mu^x\delta(x-k)}{x!}\,dx\\ &= \exp(-\mu)\sum_{k=0}^{\infty}\frac{\mu^k}{k!} = \exp(-\mu)\exp(\mu) = 1,\end{aligned}$$

as (20.1) requires.

Compared to the exponential distribution (§ 20.8.2), the Poisson distribution serves to model for example the number of customers to arrive at a retail establishment during the next hour.

20.8.4 The Rayleigh distribution

The *Rayleigh distribution* is a generalization of the normal distribution for position in a plane. Let each of the x and y coordinates be drawn independently from a normal distribution of zero mean and unit standard deviation, such that

$$\begin{aligned} dP &\equiv [\Omega(x)\,dx]\,[\Omega(y)\,dy] \\ &= \frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right)dx\,dy \\ &= \frac{1}{2\pi}\exp\left(-\frac{\rho^2}{2}\right)\rho\,d\rho\,d\phi, \end{aligned}$$

whence

$$\begin{aligned} P_{ba} &\equiv \int_{\phi=-\pi}^{\pi}\int_{\rho=a}^{b} dP \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{a}^{b}\exp\left(-\frac{\rho^2}{2}\right)\rho\,d\rho\,d\phi \\ &= \int_{a}^{b}\exp\left(-\frac{\rho^2}{2}\right)\rho\,d\rho, \end{aligned}$$

which implies the distribution

$$f(\rho) = u(\rho)\rho\exp\left(-\frac{\rho^2}{2}\right). \tag{20.33}$$

This is the Rayleigh distribution. That it is a proper distribution according to (20.1) is proved by evaluating the integral

$$\int_0^\infty f(\rho)\,d\rho = 1 \tag{20.34}$$

using part of the method of § 18.4. Rayleigh's CDF (20.2) and quantile (20.4) are evidently[15]

$$\begin{aligned} F(\rho) &= 1 - \exp\left(-\frac{\rho^2}{2}\right), \quad \rho \ge 0; \\ F^{-1}(v) &= \sqrt{-2\ln(1-v)}. \end{aligned} \tag{20.35}$$

[15] [125, § 5.2]

The Rayleigh distribution models among others the distance ρ by which a missile might miss its target.

Incidentally, there is nothing in the mathematics to favor any particular value of ϕ over another, ϕ being the azimuth at which the missile misses, for the integrand $\exp(-\rho^2/2)\rho\,d\rho\,d\phi$ includes no ϕ. The azimuth ϕ must by symmetry therefore be uniformly distributed.

Rayleigh's mean and standard deviation are computed via (20.8) to be

$$\begin{aligned}\mu &= \frac{\sqrt{2\pi}}{2},\\ \sigma^2 &= 2 - \frac{2\pi}{4}.\end{aligned} \tag{20.36}$$

by

$$\mu = \int_0^\infty \rho^2 \exp\left(-\frac{\rho^2}{2}\right) d\rho = \frac{\sqrt{2\pi}}{2}$$

(compare eqn. 20.18, observing however that the present integral integrates over only half the domain) and

$$\begin{aligned}\sigma^2 &= \int_0^\infty \left(\rho - \frac{\sqrt{2\pi}}{2}\right)^2 \rho \exp\left(-\frac{\rho^2}{2}\right) d\rho\\
&= \int_0^\infty \rho^3 \exp\left(-\frac{\rho^2}{2}\right) d\rho\\
&\qquad - \sqrt{2\pi}\int_0^\infty \rho^2 \exp\left(-\frac{\rho^2}{2}\right) d\rho\\
&\qquad + \frac{2\pi}{4}\int_0^\infty \rho \exp\left(-\frac{\rho^2}{2}\right) d\rho\\
&= \int_0^\infty \rho^3 \exp\left(-\frac{\rho^2}{2}\right) d\rho - \frac{2\pi}{2} + \frac{2\pi}{4}\\
&= -\int_{\rho=0}^\infty \rho^2\, d\left[\exp\left(-\frac{\rho^2}{2}\right)\right] - \frac{2\pi}{4}\\
&= -\rho^2 \exp\left(-\frac{\rho^2}{2}\right)\Big|_0^\infty + \int_{\rho=0}^\infty \exp\left(-\frac{\rho^2}{2}\right) d\left[\rho^2\right] - \frac{2\pi}{4}\\
&= 0 + 2\int_0^\infty \rho \exp\left(-\frac{\rho^2}{2}\right) d\rho - \frac{2\pi}{4} = 2 - \frac{2\pi}{4}.\end{aligned}$$

20.8.5 The Maxwell distribution

The *Maxwell distribution* extends the Rayleigh from two to three dimensions. Maxwell's derivation closely resembles Rayleigh's, with the difference that Maxwell uses all three of x, y and z and then transforms to spherical rather than cylindrical coordinates. The distribution which results, the Maxwell distribution, is

$$f(r) = \frac{2r^2}{\sqrt{2\pi}} \exp\left(-\frac{r^2}{2}\right), \quad r \geq 0, \tag{20.37}$$

which models among others the speed at which an air molecule might travel.[16]

20.9 The Box-Muller transformation

The quantiles (20.31) and (20.35) imply easy conversions from the uniform distribution to the exponential and Rayleigh. Unfortunately, we lack a quantile formula for the normal distribution. However, we can still convert uniform to normal by way of Rayleigh as follows.

Section 20.8.4 has associated the Rayleigh distribution with the distance ρ by which a missile misses its target, the x and y coordinates of the missile's impact each being normally distributed over equal standard deviations. Section 20.8.4 has further drawn out the uniform distribution of the impact's azimuth ϕ. Because we know Rayleigh's quantiles, we are able to convert a pair of instances u and v of a uniformly distributed random variable to Rayleigh's distance and azimuth by[17]

$$\begin{aligned} \rho &= \sqrt{-2\ln(1-u)}, \\ \phi &= (2\pi)\left(v - \frac{1}{2}\right). \end{aligned} \tag{20.38}$$

But for the reason just given,

$$\begin{aligned} x &= \rho\cos\phi, \\ y &= \rho\sin\phi, \end{aligned} \tag{20.39}$$

[16] [57, eqn. I:40.7]

[17] One can eliminate a little trivial arithmetic by appropriate changes of variable in (20.38) like $u' \leftarrow 1 - u$, but to do so saves little computational time and makes the derivation harder to understand. Still, the interested reader might complete the improvement as an exercise.

must then constitute two independent instances of a normally distributed random variable with $\mu = 0$ and $\sigma = 1$. Evidently, though we lack an easy way to convert a single uniform instance to a single normal instance, we can convert a *pair* of uniform instances to a pair of normal instances. Equations (20.38) and (20.39) are the *Box-Muller transformation.*[18]

20.10 The normal CDF at large arguments

The Taylor series (20.20) in theory calculates the normal CDF $F_\Omega(x)$, an entire function, for any argument x. In practice however, consider the Taylor series

$$1 - F_\Omega(6) \approx -\mathrm{0x0.8000} + \mathrm{0x2.64C6} - \mathrm{0xE.5CA7} + \mathrm{0x4D.8DEC} - \cdots$$

Not promising, is it? Using a computer's standard, `double`-type floating-point arithmetic, this calculation fails, swamped by rounding error.

One can always calculate in greater precision,[19] of course, asking the computer to carry extra bits; and, actually, this is not necessarily a bad approach. There remain however several reasons one might prefer a more efficient formula.

- One might wish to evaluate the CDF at thousands or millions of points, not just one. At some scale, even with a computer, the calculation grows expensive.
- One might wish to evaluate the CDF on a low-power "embedded device."
- One might need to evaluate the CDF under a severe time constraint measured in microseconds, as in aircraft control.
- Hard though it might be for some to imagine, one might actually wish to evaluate the CDF with a pencil! Or with a slide rule. (Besides that one might not have a suitable electronic computer conveniently at hand, that electronic computers will never again be scarce is a proposition whose probability the author is not prepared to evaluate.)
- The mathematical method by which the more efficient formula is derived is instructive.[20]

[18] [28][178]

[19] [138]

[20] Such methods lead one to wonder how much useful mathematics our civilization should have forgone had hardy mathematical pioneers like Leonhard Euler (1707–1783) and Carl Friedrich Gauss (1777–1855) had computers to lean on.

- One might regard a prudent measure of elegance, even in applications, to be its own reward.

Here is the method.[21] Beginning from

$$
\begin{aligned}
1 - F_\Omega(x) &= \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{\tau^2}{2}\right) d\tau \\
&= \frac{1}{\sqrt{2\pi}} \left\{ -\int_{\tau=x}^\infty \frac{d\left[e^{-\tau^2/2}\right]}{\tau} \right\}
\end{aligned}
$$

and integrating by parts,

$$
\begin{aligned}
1 - F_\Omega(x) &= \frac{1}{\sqrt{2\pi}} \left\{ \frac{e^{-x^2/2}}{x} - \int_x^\infty \frac{e^{-\tau^2/2}\,d\tau}{\tau^2} \right\} \\
&= \frac{1}{\sqrt{2\pi}} \left\{ \frac{e^{-x^2/2}}{x} + \int_{\tau=x}^\infty \frac{d\left[e^{-\tau^2/2}\right]}{\tau^3} \right\}.
\end{aligned}
$$

Integrating by parts again,

$$
\begin{aligned}
1 - F_\Omega(x) &= \frac{1}{\sqrt{2\pi}} \left\{ \frac{e^{-x^2/2}}{x} - \frac{e^{-x^2/2}}{x^3} + 3\int_x^\infty \frac{e^{-\tau^2/2}\,d\tau}{\tau^4} \right\} \\
&= \frac{1}{\sqrt{2\pi}} \left\{ \frac{e^{-x^2/2}}{x} - \frac{e^{-x^2/2}}{x^3} - 3\int_{\tau=x}^\infty \frac{d\left[e^{-\tau^2/2}\right]}{\tau^5} \right\}.
\end{aligned}
$$

Integrating by parts repeatedly,

$$
\begin{aligned}
1 - F_\Omega(x) = \frac{1}{\sqrt{2\pi}} \Bigg\{ & \frac{e^{-x^2/2}}{x} - \frac{e^{-x^2/2}}{x^3} + \frac{3e^{-x^2/2}}{x^5} - \cdots \\
& + \frac{(-)^{n-1}(2n-3)!!e^{-x^2/2}}{x^{2n-1}} \\
& + (-)^n(2n-1)!! \int_x^\infty \frac{e^{-\tau^2/2}\,d\tau}{\tau^{2n}} \Bigg\},
\end{aligned}
$$

[21] [107, § 2.2]

in which the convenient notation

$$
\begin{aligned}
m!! &\equiv \begin{cases} \prod_{j=1}^{(m+1)/2}(2j-1) = (m)(m-2)\cdots(5)(3)(1) & \text{for odd } m, \\ \prod_{j=1}^{m/2}(2j) \qquad\quad\;\; = (m)(m-2)\cdots(6)(4)(2) & \text{for even } m, \end{cases} \\
0!! &= (-1)!! = 1,
\end{aligned}
\tag{20.40}
$$

is introduced.[22] The last expression for $1 - F_\Omega(x)$ is better written,

$$
\begin{aligned}
1 - F_\Omega(x) &= \frac{\Omega(x)}{x}[S_n(x) + R_n(x)], \qquad (20.41)\\
S_n(x) &\equiv \sum_{k=0}^{n-1}\left[\prod_{j=1}^{k}\frac{2j-1}{-x^2}\right] = \sum_{k=0}^{n-1}\frac{(-)^k(2k-1)!!}{x^{2k}}, \\
R_n(x) &\equiv (-)^n(2n-1)!!x\int_x^\infty \frac{e^{(x^2-\tau^2)/2}\,d\tau}{\tau^{2n}}.
\end{aligned}
$$

The series $S_n(x)$ is an *asymptotic series,* also called a *semiconvergent series.*[23] So long as $x \gg 1$, the series' first several terms successively shrink in magnitude but, no matter how great x is, the terms eventually insist on growing again, without limit. Unlike a Taylor series, $S_\infty(x)$ diverges for all x.

Fortunately, nothing requires us to let $n \to \infty$, and we remain free to choose n strategically as we like—for instance to exclude from S_n the series' least term in magnitude and all the terms following. So excluding leaves us with the problem of evaluating the integral R_n, but see:

$$
\begin{aligned}
|R_n(x)| &\le (2n-1)!!\,|x|\int_x^\infty\left|\frac{e^{(x^2-\tau^2)/2}\,d\tau}{\tau^{2n}}\right| \\
&\le \frac{(2n-1)!!}{|x|^{2n}}\int_x^\infty\left|e^{(x^2-\tau^2)/2}\tau\,d\tau\right|
\end{aligned}
$$

because $|x| \le |\tau|$, so $|x|^{2n+1} \le |\tau|^{2n+1}$. Changing $\xi^2 \leftarrow \tau^2 - x^2$, whereby $\xi\,d\xi = \tau\,d\tau$,

$$
|R_n(x)| \le \frac{(2n-1)!!}{|x|^{2n}}\int_0^\infty\left|e^{-\xi^2/2}\xi\,d\xi\right|.
$$

[22] [3, Exercise 2.2.15]
[23] [3, § 1.4.1]

Using (20.33) and (20.34),

$$|R_n(x)| \leq \frac{(2n-1)!!}{|x|^{2n}}, \quad \Im(x) = 0,\ \Re(x) > 0, \tag{20.42}$$

which in view of (20.41) has that the magnitude $|R_n|$ of the error due to truncating the series after n terms does not exceed the magnitude of the first omitted term. Equation (20.41) thus provides the efficient means we have sought to estimate the CDF accurately for large arguments.

20.11 Asymptotic series

Section 20.10 has incidentally introduced the *asymptotic series* and has shown how to treat it.

Asymptotic series are strange. They diverge, but only after approaching a sum of interest. Some asymptotic series approach the sum of interest quite closely, and moreover do so in such a way that the closeness—that is, the error in the sum—can with sufficient effort be quantified. The error in the sum of the asymptotic series of § 20.10 has been found not to exceed the magnitude of the first omitted term; and though one may have to prove it specially for each such series, various series one encounters in practice tend to respect bounds of the same kind.

As § 20.10 has noted, asymptotic series are sometimes alternately called *semiconvergent series.*[24]

An ordinary, convergent series is usually preferable to an asymptotic series, of course, especially in the subdomain near the convergent series' expansion point (§ 8.2). However, a convergent series is not always available; and, even when it is, its expansion point may lie so distant that the series becomes numerically impractical to total.

An asymptotic series can fill the gap.

Aside from whatever practical applications an asymptotic series can fill, this writer finds the topic of asymptotic series fascinating. The topic is curious, is it not? How can a *divergent series* reach a definite total? The answer seems to be: it cannot reach a definite total but can draw arbitrarily close to one. In (20.41) and (20.42) for example, the larger the argument, the closer the draw. It is a paradox yet, surprisingly, it works.

Asymptotic series arise in the study and application of special functions, including (as we have seen) the $\Omega(\cdot)$ of the present chapter. For this rea-

[24]See footnote 23.

son and maybe others, the applied mathematician will exercise and exploit asymptotic series from time to time.

20.12 The normal quantile

Though no straightforward quantile formula to satisfy (20.4) for the normal distribution seems to be known, nothing prevents one from calculating the quantile via the Newton-Raphson iteration (4.30)[25]

$$\begin{aligned} x_{k+1} &= x_k - \frac{F_\Omega(x_k) - v}{\Omega(x_k)}, \\ F_\Omega^{-1}(v) &= \lim_{k\to\infty} x_k, \\ x_0 &= 0, \end{aligned} \tag{20.43}$$

where $F_\Omega(x)$ is as given by (20.20) and/or (20.41) and where $\Omega(x)$ is, as usual, as given by (20.17). The shape of the normal CDF as seen in Fig. 20.1 on page 688—curving downward traveling right from $x = 0$, upward when traveling left—evidently guarantees convergence per Fig. 4.6, page 125.

In the large-argument limit,

$$\begin{aligned} 1 - v &\ll 1, \\ x &\gg 1; \end{aligned}$$

so, according to (20.41),

$$F_\Omega(x) \approx 1 - \frac{\Omega(x)}{x}\left(1 - \frac{1}{x^2} + \cdots\right).$$

[25]When implementing numerical algorithms like these on the computer one should do it intelligently. For example, if $F_\Omega(x_k)$ and v are both likely to be close to 1, then do not ask the computer to calculate and/or store these quantities. Rather, ask it to calculate and/or store $1 - F_\Omega(x_k)$ and $1 - v$. Then, when (20.43) instructs you to calculate a quantity like $F_\Omega(x_k) - v$, let the computer instead calculate $[1-v] - [1 - F_\Omega(x_k)]$, which is arithmetically no different but numerically, on the computer, much more precise.

Substituting this into (20.43) yields, by successive steps,

$$\begin{aligned}
x_{k+1} &\approx x_k - \frac{1}{\Omega(x_k)}\left[1 - v - \frac{\Omega(x_k)}{x_k}\left(1 - \frac{1}{x_k^2} + \cdots\right)\right] \\
&\approx x_k - \frac{1-v}{\Omega(x_k)} + \frac{1}{x_k} - \frac{1}{x_k^3} + \cdots \\
&\approx x_k - \frac{\left(\sqrt{2\pi}\right)(1-v)}{1 - x_k^2/2 + \cdots} + \frac{1}{x_k} - \frac{1}{x_k^3} + \cdots \\
&\approx x_k - \left(\sqrt{2\pi}\right)\left(1-v\right)\left(1 + \frac{x_k^2}{2} + \cdots\right) + \frac{1}{x_k} - \frac{1}{x_k^3} + \cdots \\
&\approx x_k - \left(\sqrt{2\pi}\right)\left(1-v\right) + \frac{1}{x_k} + \cdots,
\end{aligned}$$

suggesting somewhat lazy, but usually acceptable convergence in domains of typical interest (the convergence might be unacceptable if, for example, $x >$ 0x40, but the writer has never encountered an application of the normal distribution $\Omega[x]$ or its incidents at such large values of x). If unacceptable, various stratagems might be tried to accelerate the Newton-Raphson, or—if you have no need to impress anyone with the pure elegance of your technique but only want the right answer reasonably fast—you might just search for the root in the naïve way, trying $F_\Omega(2^0)$, $F_\Omega(2^1)$, $F_\Omega(2^2)$ and so on until identifying a bracket $F_\Omega(2^{k-1}) < v \le F_\Omega(2^k)$; then dividing the bracket in half, then in half again, then again and again until satisfied with the accuracy thus achieved, or until the bracket were strait enough for you to set x_0 to the bracket's lower (not upper) limit and to switch over to (20.43) which performs well when it starts close enough to the root. In truth, though not always stylish, the normal quantile of a real argument is relatively quick, easy and accurate to calculate once you have (20.17), (20.20) and (20.41) in hand, even when the performance of (20.43) might not quite suit. You only must remain a little flexible as to the choice of technique.[26]

[26] See also § 8.10.4.

Chapter 21

The gamma function

This chapter studies a special function called the *gamma function.*

21.1 Special functions

The gamma function being a special function, the chapter might begin by telling what a *special function* is. However, a definition of the term is hard to find. N. N. Lebedev begins chapter 1 of his book on special functions,

> One of the simplest and most important special functions is the *gamma function.*[1] [107]

The gamma is a special function and is simple and important, reveals Lebedev; but what a special function is, or why the gamma should be one, is not stated. Larry C. Andrews begins the preface of *his* book on special functions,

> Modern engineering and physics applications demand a ... thorough knowledge of ... the basic properties of *special functions.*[2]

The italics suggest that Andrews might next define the term, but instead he continues,

> These functions commonly arise in such areas of application as heat conduction, communications systems, electro-optics, ... [3]

[1]Emphasis in the original.
[2]Emphasis in the original.

In [1], Abramowitz and Stegun deliver two early chapters named "Elementary analytical methods" and "Elementary transcendental functions" before launching into a series of nineteen chapters on special functions (exponential integral, gamma, error/Fresnel, Legendre, Bessel-integer, Bessel-fractional, Bessel integral, Struve, confluent hypergeometric, Coulomb, hypergeometric, Jacobian elliptic/theta, elliptic integral, Weierstrass elliptic, parabolic cylinder, Mathieu, spheroidal wave, orthogonal polynomial and Bernoulli/ Euler/Riemann[3]), by which one may infer that special functions are *nonelementary;* but even Abramowitz and Stegun never quite seem to say what a special function is. Gradshteyn and Ryzhik agree with Abramowitz and Stegun that special functions are nonelementary:

> First, we have the elementary functions: the function $f(x) = x$; the exponential function; the hyperbolic functions; the trigonometric functions; the logarithmic function; the inverse hyperbolic functions . . . ; the inverse trigonometric functions.
>
> Then follow the special functions: elliptic integrals; elliptic functions; the logarithm . . . , exponential . . . , sine . . . and . . . cosine integral functions; probability integrals and Fresnel's integrals; the gamma function and related functions; Bessel functions; Mathieu functions; Legendre functions; orthogonal polynomials; hypergeometric functions; degenerate hypergeometric functions; parabolic cylinder functions; Meijer's and MacRobert's functions; Riemann's zeta function. [68]

No actual definition is seen here, either, though. And W. W. Bell? Bell too has written a book on special functions. In his book Bell begins chapter 1,

> Many special functions arise in the consideration of solutions of equations of the form $P(x)\, d^2y/dx^2 + Q(x)\, dy/dx + R(x)y = 0$. [18]

And so on. The desultory collection of quotations this paragraph has assembled is no exhaustive survey, of course, but *special function* nevertheless appears to be a term specialists[4] were reluctant to define.

Eric W. Weisstein, like your author a nonspecialist, does define the term:

> A special function is a function (usually named after an early investigator of its properties) having a particular use in mathematical physics or some other branch of mathematics. Prominent

[3] The book you are reading will treat only a few of the topics of Abramowitz's and Stegun's nineteen chapters.

[4] Neat word. Take "specialist" either way.

> examples include the gamma function, hypergeometric function, Whittaker function, and Meijer G-function. [177]

Such a nonspecialist's definition however, presenting itself chiefly in terms of examples, differs from the *specialists' nondefinitions* less than one might like.

If specialists like Lebedev, Andrews, Abramowitz, Stegun, Gradshteyn, Ryzhik and Bell cannot or will not define the term *special function* more precisely than they have, then perhaps the book you are reading should leave the question in the form in which the specialists have left it. Apparently, an applied mathematician is to recognize a special function when he meets it! The gamma function is one at any rate, by common consent.

Besides the gamma, we have already met two other special functions in this book: the sine integral of § 17.6; and the cumulative normal distribution of §§ 20.5.2 and 20.10. Furthermore, § 20.10 and 20.11 have introduced asymptotic series, chiefly a special-functions topic.

Special functions tend to have in common that they

- solve differential equations, definite integrals or integral equations (§ 21.9) of applicationary interest; but

- cannot readily be expressed in terms of polynomials of finite numbers of terms, nor in terms of combinations of such polynomials with ratios, exponentials, trigonometrics, logarithms and/or inverse trigonometrics; and

- may want asymptotic series to calculate their values;

but otherwise special functions are various.

The fascination of special functions to the scientist and engineer lies in how gracefully they analyze otherwise intractable physical models; in how reluctantly they yield their mathematical secrets; in how readily they conform to unexpected applications; in how often they connect seemingly unrelated phenomena; and in that, the more intrepidly one explores their realm, the more disquietly one feels that one had barely penetrated the realm's frontier. The topic of special functions seems inexhaustible. We surely will not begin to exhaust the topic in this book; yet, even so, useful results will start to flow from our study almost at once.

The chapter you are reading mainly regards the gamma function.

21.2 The definite integral representation

Consider the definite integral

$$\int_{t_1}^{t_2} \tau^{a-1}\, d\tau,$$

slightly perplexing in that it diverges if $t_1 \to 0^+$ and $a \le 0$, yet also diverges if $t_2 \to \infty$ and $a \ge 0$; and more perplexing in that it diverges regardless of the value of a if both $t_1 \to 0^+$ and $t_2 \to \infty$, a frequent case even in applied work. To coërce convergence, various tactics might be tried, among which is to damp the integrand[5] by

$$\int_{t_1}^{t_2} e^{-\tau}\tau^{a-1}\, d\tau.$$

The damped integral converges even if $t_1 \to 0^+$ and $t_2 \to \infty$ as long as a is positive. When $z \leftarrow a$ is changed to support complex exponents, the definite integral[6] that results,

$$\Gamma(z) \equiv \int_0^\infty e^{-\tau}\tau^{z-1}\, d\tau, \quad \Re(z) > 0, \tag{21.1}$$

has been found to be sufficiently interesting to merit a name. Its name is the *gamma function.* It is this chapter's chief subject.

---

[5]Damping the integrand is a practical technique to force a nonconvergent integral to converge [30]. Whether this section's integral is the right kind of integral to damp can be debated (the book prints the damping attempt because the attempt incidentally discovers the gamma function) but one might damp other integrals by

$$\lim_{\epsilon\to 0^+} \int_0^\infty e^{-\epsilon^n \alpha\tau} f(\tau)\, d\tau$$

or

$$\lim_{\epsilon\to 0^+} \int_{-\infty}^\infty e^{-\epsilon^n \alpha\tau^2/2} f(\tau)\, d\tau$$

or the like, the n being a small nonnegative integer (typically 0, 1 or 2) and the α being a positive real number. The present section tries damping by $e^{-\epsilon^n\alpha\tau}$ with $n = 0$ and $\alpha = 1$. (If unsure what the n is for, then see for example § 19.5 in which $n = 2$ is used.)

[6][107, eqn. 1.1.1][3, eqn. 2.5][1, eqn. 6.1.1][144, eqn. 2]

21.3 Relationship to the factorial

Expressing $\Gamma(z+1)$ according to (21.1) and using (9.11) to integrate by parts[7],

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty e^{-\tau}\tau^z\,d\tau \\ &= -\int_0^\infty \tau^z\,d[e^{-\tau}] \\ &= -e^{-\tau}\tau^z\big|_{\tau=0}^{\infty} + \int_0^\infty e^{-\tau}\,d[\tau^z] \\ &= 0 + z\int_0^\infty e^{-\tau}\tau^{z-1}\,d\tau.\end{aligned}$$

That is,[8]

$$\Gamma(z+1) = z\Gamma(z). \tag{21.2}$$

According to (21.1),[9]

$$\Gamma(1) = 1, \tag{21.3}$$

so by induction on (21.2) we conclude that[10]

$$n! = \Gamma(n+1). \tag{21.4}$$

The recursion (21.2) incidentally affords a means to calculate $\Gamma(z)$ for $\Re(z) \le 0$—although, according to (21.2) or (21.4), $\Gamma(z)$ diverges[11] for $z < 0$ if $z \in \mathbb{Z}$.

21.4 Half-integral arguments

Using technique[12] like that of § 18.4, changing $u^2 \leftarrow \tau$ and $x \leftarrow z$ in (21.1),

$$\Gamma(x) = 2\int_0^\infty e^{-u^2}u^{2x-1}\,du.$$

Similarly,

$$\Gamma(y) = 2\int_0^\infty e^{-v^2}v^{2y-1}\,dv.$$

[7] [107, § 1.2]
[8] [107, eqn. 1.2.1][3, eqn. 2.3][1, eqn. 6.1.15]
[9] [3, eqn. 2.2]
[10] [3, eqn. 2.4][1, eqn. 6.1.6]
[11] Section 21.7 will address the gamma function's analyticity and poles.
[12] [3, § 2.2.1]

The product of the last two equations has that

$$\begin{aligned}
\Gamma(x)\Gamma(y) &= 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}u^{2x-1}v^{2y-1}\,du\,dv \\
&= 4\int_0^{2\pi/4}\int_0^\infty e^{-\rho^2}(\rho\cos\phi)^{2x-1}(\rho\sin\phi)^{2y-1}\rho\,d\rho\,d\phi \\
&= 4\int_0^{2\pi/4}\int_0^\infty e^{-\rho^2}\rho^{2(x+y)-1}\cos^{2x-1}\phi\sin^{2y-1}\phi\,d\rho\,d\phi.
\end{aligned}$$

Evaluating at $x = 1/2$ and $y = 1/2$,

$$\Gamma^2\left(\frac{1}{2}\right) = 4\int_0^{2\pi/4}\int_0^\infty e^{-\rho^2}\rho\,d\rho\,d\phi = 2\pi\int_0^\infty e^{-\rho^2}\rho\,d\rho = -\pi e^{-\rho^2}\Big|_0^\infty = \pi.$$

That is,[13] insofar as (21.1) makes $\Gamma(z)$ to be positive for all real, positive z,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{21.5}$$

Equations (21.2) and (21.5) together determine $\Gamma(z)$ for all half-integral $z = n + 1/2$, $n \in \mathbb{Z}$. For example, $\Gamma(3/2) = (1/2)\Gamma(1/2) = (\sqrt{\pi})/2$ and $\Gamma(-1/2) = \Gamma(1/2)/(-1/2) = -2\sqrt{\pi}$.

21.5 Numerical evaluation

Not every method used to expand elementary functions works on a special function like the gamma but one method that almost works is to expand the exponential in (21.1) via Table 8.1, obtaining the form

$$\Gamma(z) = \int_0^\infty \sum_{k=0}^\infty \frac{(-)^k\tau^{k+z-1}}{k!}\,d\tau.$$

Here, one would like to swap the $\int$ integration and $\sum$ summation signs to integrate each term but, unfortunately, the term-by-term integrals diverge.

Better is to split (21.1)'s integral at $\tau = T$ into two domains as

$$\Gamma(z) = \int_0^T e^{-\tau}\tau^{z-1}\,d\tau + \int_T^\infty e^{-\tau}\tau^{z-1}\,d\tau,\quad \Im(T) = 0,\ \Re(T) > 0, \tag{21.6}$$

[13] [107, eqn. 1.2.5][3, eqn. 2.23][1, eqn. 6.1.8]

the T being a nonnegative real number one can choose at discretion—the larger, the numerically harder (a computer needing wider registers with more bits) but the more accurate the calculation.

The leftward integral now supports the $\int$-$\sum$ swap we had wanted earlier:

$$\begin{aligned}\Gamma(z) &= \sum_{k=0}^{\infty}\int_0^T \frac{(-)^k\tau^{k+z-1}}{k!}\,d\tau + \int_T^\infty e^{-\tau}\tau^{z-1}\,d\tau \\ &= \sum_{k=0}^{\infty}\frac{(-)^k T^{k+z}}{k!(k+z)} + \int_T^\infty e^{-\tau}\tau^{z-1}\,d\tau. \end{aligned} \tag{21.7}$$

The rightward integral is

$$I = \int_T^\infty e^{-\tau}\tau^{z-1}\,d\tau. \tag{21.8}$$

If $0 < \Re(z) < 1$,

$$|I| < \left|T^{z-1}\right| \int_T^\infty e^{-\tau}\,d\tau.$$

Evaluating,

$$|I| < \left|T^{z-1}\right| e^{-T} \text{ for } 0 < \Re(z) < 1. \tag{21.9}$$

If $\Im(z) = 0$ and $0 < z < 1$ then (21.9) implies that

$$0 < I < T^{z-1}e^{-T},$$

since the integrand in (21.8) is never negative in the real-valued case. Indeed, a yet tighter bound can be established in the real-valued case by observing the factor τ^{z-1} in (21.8) and estimating, for τ in the neighborhood of T, that

$$\tau^{z-1} \approx T^{z-1}e^{(\alpha)(T-\tau)}$$

the estimate evidently being exact at $\tau = T$ regardless of the value of α, the α being an arbitrary parameter with a value to be chosen momentarily. Differentiating the estimate with respect to τ,

$$(z-1)\tau^{z-2} \approx -\alpha T^{z-1}e^{(\alpha)(T-\tau)}.$$

Preferring the last to be exact at $\tau = T$, we require that

$$(z-1)T^{z-2} = -\alpha T^{z-1}e^{(\alpha)(T-T)}$$

or, solving, that

$$\alpha = -\frac{z-1}{T},$$

which lets us express the estimate—and here we refer to the original estimate rather than to its derivative—as

$$\tau^{z-1} \approx T^{z-1} e^{(z-1)(\tau/T-1)}.$$

For a reason that will soon grow clear, it would be convenient in view of the last line if

$$\begin{gathered} \tau^{z-1} > T^{z-1} e^{(z-1)(\tau/T-1)} > 0 \\ \text{for } 0 < T < \tau,\ \Im(T) = 0,\ \Im(\tau) = 0, \\ 0 < z < 1,\ \Im(z) = 0, \end{gathered} \tag{21.10}$$

or, rearranging factors, if

$$\left(\frac{\tau}{T}\right)^{z-1} > e^{(z-1)(\tau/T-1)} > 0$$

or, raising both sides to the $1/(z-1)$ power (where $1/[z-1] < 0$ because $z < 1$), if

$$0 < \frac{\tau}{T} < e^{\tau/T-1}$$

or, expressing both sides in terms of the exponent, if

$$0 < 1 + \left(\frac{\tau}{T} - 1\right) < e^{\tau/T-1}$$

over the domain (21.10) stipulates. The quantity $\tau/T-1$ being positive over the domain, a Taylor expansion of the last inequality's exponential proves the inequality to be true and, the steps leading to the last inequality being reversible, thereby verifies (21.10).

Applying (21.10) to (21.8),

$$I > \int_T^\infty e^{-\tau} T^{z-1} e^{(z-1)(\tau/T-1)}\, d\tau = e^{-(z-1)} T^{z-1} \int_T^\infty e^{[(z-1)/T-1]\tau}\, d\tau.$$

Unlike the integral of (21.8), the last integral is one we know how to evaluate. Evaluating it,

$$I > \frac{e^{-(z-1)} T^{z-1}}{(z-1)/T-1} e^{[(z-1)/T-1]\tau} \bigg|_{\tau=T}^{\infty} = -\frac{e^{-(z-1)} T^{z-1}}{(z-1)/T-1} e^{[(z-1)/T-1]T}.$$

That is,

$$I > \frac{T^{z-1} e^{-T}}{1-(z-1)/T}.$$

Combining the last inequality with the unnumbered inequality following (21.9), we conclude that

$$\frac{T^{z-1}e^{-T}}{1-(z-1)/T} < I < T^{z-1}e^{-T} \quad \text{for } 0 < T,\ \Im(T) = 0,\ 0 < z < 1,\ \Im(z) = 0; \tag{21.11}$$

and by similar reasoning[14] that

$$T^{z-1}e^{-T} < I < \frac{T^{z-1}e^{-T}}{1-(z-1)/T} \quad \text{for } 0 < T,\ \Im(T) = 0,\ 1 < z,\ \Im(z) = 0. \tag{21.12}$$

Finally, using (21.8) to condense the representation of (21.7),

$$\Gamma(z) = \sum_{k=0}^{\infty} \frac{(-)^k T^{k+z}}{k!(k+z)} + I, \tag{21.13}$$

with either (21.9), in the complex case, or (21.11) or (21.12), in the real case, providing a bounded estimate of I. Equation (21.7) numerically evaluates $\Gamma(z)$.

Several other methods to evaluate $\Gamma(z)$ are known, incidentally, as for instance in [1, chapter 6].

Figure 21.1 plots the gamma function per (21.13), with (21.2), over the real domain.[15]

21.6 Reflection

If evaluating the gamma function at a certain value of z is inconvenient, the gamma function's *reflection formula,*[16]

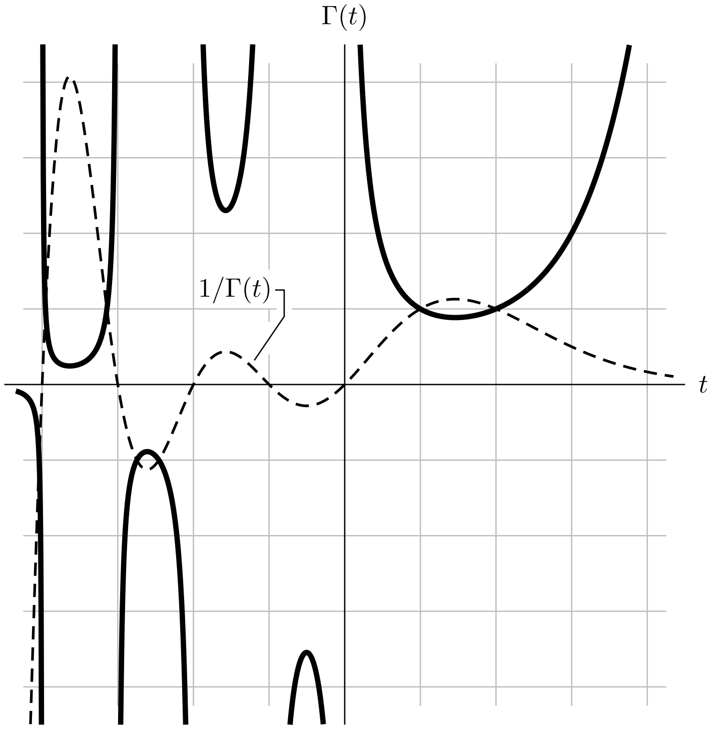
$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \tag{21.14}$$

[14] To reach (21.12), require in (21.10) that $\tau^{z-1} < T^{z-1}e^{(z-1)(\tau/T-1)}$.

[15] Similar is [1, Fig. 6.1] but because that figure predated computer plotting, the present figure is more accurate.

[16] [1, eqn. 6.1.17][107, eqn. 1.2.2][3, eqn. 2.31]

Figure 21.1: The gamma function.



can help. The formula is proved for $0 < \Re(z) < 1$ by writing per (21.1) that[17]

$$\begin{aligned}\Gamma(z)\Gamma(1-z) &= \int_0^\infty e^{-\tau}\tau^{z-1}\,d\tau \int_0^\infty e^{-\sigma}\sigma^{-z}\,d\sigma \\ &= \int_0^\infty \frac{e^{-\sigma}}{\sigma}\left[\int_0^\infty e^{-(\sigma)(\tau/\sigma)}\left(\frac{\tau}{\sigma}\right)^{z-1}d\tau\right]d\sigma,\end{aligned}$$

in which the order of integration does not matter[18] because the real parts of both $z-1$ and $-z$ lie between -1 and 0. Changing $v \leftarrow \tau/\sigma$ within the inner integral,

$$\begin{aligned}\Gamma(z)\Gamma(1-z) &= \int_0^\infty e^{-\sigma}\left[\int_0^\infty e^{-\sigma v}v^{z-1}\,dv\right]d\sigma \\ &= \int_0^\infty \left[\int_0^\infty e^{-(\sigma)(1+v)}v^{z-1}\,dv\right]d\sigma.\end{aligned}$$

Again swapping the order,

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty v^{z-1}\left[\int_0^\infty e^{-(\sigma)(1+v)}\,d\sigma\right]dv.$$

Changing $u \leftarrow (\sigma)(1+v)$ within the inner integral,

$$\begin{aligned}\Gamma(z)\Gamma(1-z) &= \int_0^\infty \frac{v^{z-1}}{1+v}\left[\int_0^\infty e^{-u}\,du\right]dv \\ &= \int_0^\infty \frac{v^{z-1}}{1+v}[1]\,dv = \int_0^\infty \frac{v^{z-1}\,dv}{1+v}.\end{aligned}$$

The last integral is perhaps nonobvious but, fortunately, if one has a sharp memory regarding integrals earlier worked (and an applied mathematician ought to cultivate such a memory to the extent to which he can), then one may recall (9.18) which—after the identity of Table 3.1 that $\sin(\phi \pm \pi) = -\sin\phi$ is applied—yields (21.14) as sought.

The recurrence (21.2) extends the reflection (21.14) to all z (except, naturally, to the nonanalytic points at $z = 0, -1, -2, -3, \ldots$). For example,[19] $\Gamma(0\mathrm{xA}/3)\Gamma(-7/3) = [\Gamma(1/3)(1/3)(4/3)(7/3)][\Gamma(2/3)/(-1/3)(-4/3)$

[17] [165, §§ 1.86 and 3.123][107, § 1.2]

[18] As the book has several times noted, justifications of convergence generally interest the professional mathematician more than they do the applicationist. Nevertheless, even at the applied level, an occasional, brief justification seems fitting as in the present instance. Reasoning similarly, the interested reader can justify convergence on the book's behalf when other, related instances arise.

[19] Formally to prove the point via induction is left as an exercise to the interested reader.

$\times\ (-7/3)] = -\Gamma(1/3)\Gamma(2/3) = -\pi/\sin[\pi(1/3)] = \pi/\sin[\pi(3+1/3)] = \pi/\sin[\pi(0xA/3)]$.

21.7 Analyticity, poles and residues

Since eqn. (21.1) makes the gamma function analytic over the domain $\Re(z) > 0$, eqn. (21.2) extends the gamma's analyticity over the whole complex plane except at the isolated points $z = 0, -1, -2, -3, \ldots$. Why? Because the fact that

$$\Gamma(z) = \frac{\Gamma(z+1)}{z},$$

the $\Gamma(z+1)$ being analytic over the domain $\Re(z) > -1$, implies that

- $\Gamma(z)$ too is analytic over the domain $\Re(z) > -1$ except at $z = 0$,
- at $z = 0$ stands a single pole (§ 2.10), and
- the residue (§ 8.8) associated with the pole is $\Gamma(z+1)|_{z=0} = 1$.

Similarly, the fact that

$$\Gamma(z) = \frac{\Gamma(z+2)}{(z)(z+1)},$$

the $\Gamma(z+2)$ being analytic over the domain $\Re(z) > -2$, implies that

- $\Gamma(z)$ is analytic over the domain $\Re(z) > -2$ except at $z = 0$ and at $z = -1$,
- at $z = -1$ stands a single pole, and
- the residue associated with the pole is $\Gamma(z+2)/z|_{z=-1} = -1$.

Generalizing,

- $\Gamma(z)$ is analytic over the whole complex plane except at $z = 0, -1, -2, -3, \ldots$,
- at each of $z = 0, -1, -2, -3, \ldots$ stands a single pole, and
- the residue at $z = -n$ is $\Gamma(z+n+1)/(z)(z+1)\cdots(z+n-2)(z+n-1)|_{z=-n} = (-n)(-n+1)\cdots(-2)(-1)$.

That is, the gamma function's residue at $z = -n$, $n \in \mathbb{Z}$, $n \geq 0$, is $(-)^n/n!$.

One can use the residue to estimate $\Gamma(z)$ in the neighborhood of a pole to be

$$\Gamma(-n+\epsilon) \approx (-)^n/n!\epsilon, \quad |\epsilon| \ll 1,\ n = 0,1,2,3,\ldots \tag{21.15}$$

For example,

$$\begin{aligned}\Gamma(-0\text{x}6.\text{FF} + i0\text{x}0.02) &= \Gamma(7 + 0\text{x}0.01 + i0\text{x}0.02)\\ &\approx (-)^5/7!(0\text{x}0.01 + i0\text{x}0.02)\\ &\approx (-0\text{x}100 + i0\text{x}200)/7!(1^2+2^2)\\ &\approx (-0\text{x}100 + i0\text{x}200)/7!5\\ &\approx -0\text{x}0.02\text{A} + i0\text{x}0.053.\end{aligned}$$

Among the consequences of this section's findings is that the *reciprocal gamma function,* $1/\Gamma(z)$, plotted in Fig. 21.1 over the real domain, is an entire function (see § 8.6). The gamma function itself is not entire, for it has poles, but it is meromorphic (see again § 8.6) because its poles are simple ones and it lacks branches and other singularities.

21.8 The digamma function

The *digamma function* is the gamma function's logarithmic derivative

$$\psi(z) \equiv \frac{d}{dz}\ln\Gamma(z) = \frac{(d/dz)\Gamma(z)}{\Gamma(z)}, \tag{21.16}$$

the logarithmic derivative as an operation having been introduced in § 4.4.10. Following Lebedev [107, § 1.3], differentiating (21.1),

$$\begin{aligned}\frac{d}{dz}\Gamma(z) &= \int_0^\infty e^{-\tau}\frac{d\left[\tau^{z-1}\right]}{dz}\,d\tau,\\ &= \int_0^\infty e^{-\tau}\frac{d\left[e^{(\ln\tau)(z-1)}\right]}{dz}\,d\tau\\ &= \int_0^\infty e^{-\tau}\tau^{z-1}\ln\tau\,d\tau\\ &= \int_0^\infty e^{-\tau}\tau^{z-1}\ln\left(\frac{\tau}{1}\right)d\tau.\end{aligned}$$

For the next step, we shall use Frullani's integral (9.31). Changing $\sigma \leftarrow \tau$ in (9.31), using $e^{-\sigma}$ for Frullani's $f(\sigma)$, and expanding $\ln(\tau/1)$ according to the result,

$$\begin{aligned}
\frac{d}{dz}\Gamma(z) &= \int_0^\infty e^{-\tau}\tau^{z-1} \int_0^\infty \frac{e^{-1\sigma} - e^{-\tau\sigma}}{\sigma}\, d\sigma\, d\tau \\
&= \int_0^\infty \frac{1}{\sigma} \int_0^\infty e^{-\tau}\tau^{z-1} \left(e^{-\sigma} - e^{-\tau\sigma}\right) d\tau\, d\sigma \\
&= \int_0^\infty \left[e^{-\sigma} \int_0^\infty e^{-\tau}\tau^{z-1}\, d\tau - \int_0^\infty e^{-(\sigma+1)\tau}\tau^{z-1}\, d\tau \right] \frac{d\sigma}{\sigma} \\
&= \int_0^\infty \left[e^{-\sigma}\Gamma(z) - \int_0^\infty e^{-(\sigma+1)\tau}\tau^{z-1}\, d\tau \right] \frac{d\sigma}{\sigma}.
\end{aligned}$$

Changing $t \leftarrow (\sigma + 1)\tau$ within the inner integral,

$$\begin{aligned}
\frac{d}{dz}\Gamma(z) &= \int_0^\infty \left[e^{-\sigma}\Gamma(z) - \frac{1}{(\sigma+1)^z} \int_0^\infty e^{-t}t^{z-1}\, dt \right] \frac{d\sigma}{\sigma} \\
&= \int_0^\infty \left[e^{-\sigma}\Gamma(z) - \frac{1}{(\sigma+1)^z}\Gamma(z) \right] \frac{d\sigma}{\sigma} \\
&= \Gamma(z) \int_0^\infty \left[e^{-\sigma} - \frac{1}{(\sigma+1)^z} \right] \frac{d\sigma}{\sigma}.
\end{aligned}$$

Changing $\tau \leftarrow \sigma$, dividing both sides by $\Gamma(z)$, and applying (21.16),

$$\psi(z) = \int_0^\infty \left[e^{-\tau} - \frac{1}{(\tau+1)^z} \right] \frac{d\tau}{\tau}, \quad \Re(z) > 0. \tag{21.17}$$

Equation (21.17) affords the digamma function an integral representation as eqn. (21.1) has afforded the gamma function one.

21.9 Integral equations (overview)

Section 21.1 has mentioned that special functions solve solve differential equations, definite integrals and integral equations. Differential equations and definite integrals are familiar from earlier chapters, but what is an integral equation?

An *integral equation*[20] is an equation for example like

$$f(z) = g(z) + \int_{-\infty}^{\infty} K(z, w) f(w)\, dw, \tag{21.18}$$

[20] [170]

in which the unknown is a function $f(z)$ integrated over a definite domain such that one cannot convert the equation to a differential equation by the expedient of differentiating it.

Integral equations are typically less tractable than differential equations. To solve them wants different techniques.

Actually, we have already met integral equations in disguise, in discretized form, in matrix notation (chapters 11 through 14) resembling

$$\mathbf{f} = \mathbf{g} + K\mathbf{f},$$

which means no more than it seems to mean; so maybe integral equations are not so strange as they look. The integral equation (21.18) is merely the matrix equation with

$$\begin{aligned} f(i\,\Delta z)\,\Delta z &\leftarrow \mathbf{f} \text{ [without the integral]}, \\ g(i\,\Delta z)\,\Delta z &\leftarrow \mathbf{g}, \\ K(i\,\Delta z, j\,\Delta w)\,\Delta z &\leftarrow K, \\ f(j\,\Delta w)\,\Delta w &\leftarrow \mathbf{f} \text{ [within the integral]} \end{aligned}$$

(the letter i representing here not the imaginary unit but rather an index as in chapter 11) and with $\Delta w = \Delta z$.

Plan

Future revisions of the book tentatively plan to add the following chapters.

17. Tensors
22. Special integrals
23. Cylinder functions
24. Legendre polynomials
25. Acceleration of convergence
26. The conjugate-gradient algorithm (maybe)
27. Logic

Future revisions also tentatively plan to develop a method to calculate the Euler-Mascheroni constant, but that method is not expected to require a chapter of its own. It should fit in the gamma function's chapter 21—though the method will presumably require a forward reference to the planned chapter on acceleration of convergence which might, consequently, be forced to move forward to precede chapter 21.

Chapter 22

Remarks

> I had a feeling once about mathematics—that I saw it all. Depth beyond depth was revealed to me—the Byss and Abyss, I saw—as one might see the transit of Venus or even the Lord Mayor's Show—a quantity passing through infinity and changing its sign from plus to minus. I saw exactly why it happened and why the tergiversation was inevitable but it was after dinner and I let it go. —Sir Winston Churchill (1874–1965) [74]

A book could tell more about derivations of applied mathematics, maybe without limit. This book ends here.

An engineer, I have long observed advantages of the applied approach to mathematics. The experience of writing the book however has let drawbacks of the applied approach impress themselves as well. Some drawbacks are nonobvious.

22.1 Frege

In 1879, the mathematician and philosopher Gottlob Frege explained,

> In apprehending a scientific truth we pass, as a rule, through various degrees of certitude. Perhaps first conjectured on the basis of an insufficient number of particular cases, a general proposition comes to be more and more securely established by being connected with other truths through chains of inferences, whether consequences are derived from it that are confirmed in some other way or whether, conversely, it is seen to be a consequence of propositions already established. Hence we can inquire, on the

> one hand, how we have gradually arrived at a given proposition and, on the other, how we can finally provide it with the most secure foundation. The first question may have to be answered differently for different persons; the second is more definite, and the answer to it is connected with the inner nature of the proposition considered. The most reliable way of carrying out a proof, obviously, is to follow pure logic, a way that, disregarding the particular characteristics of objects, depends solely on those laws upon which all knowledge rests. . . . *To prevent anything intuitive from penetrating here unnoticed,*[1] I had to bend every effort to keep the chain of emphasis free of gaps. In attempting to comply with this requirement in the strictest possible way I found the inadequacy of language to be an obstacle; no matter how unwieldy the expressions I was ready to accept, I was less and less able, as the relations became more and more complex, to attain the precision that my purpose required. This deficiency led me to the idea of [an] ideography [whose] first purpose . . . is to provide us with the most reliable test of the validity of a chain of inferences and to point out every presupposition that tries to sneak in unnoticed, so that its origin can be investigated. [61, Preface]

Frege's sentiments are poignant in that, like Gödel and unlike Hilbert, Frege can fairly be described as a Platonist. (Witness Frege's words in a later book: "In arithmetic we are not concerned with objects which we come to know as something alien from without through the medium of the senses, but with objects given directly to our reason and, as its nearest kin, utterly transparent to it. And yet, or rather for that very reason, these objects are not subjective fantasies. There is nothing more objective than the laws of arithmetic."[2]) To Frege, unlike to Hume or even to Kant, Platonic realism affords all the more cause to distrust merely human processes of inference in mathematical matters.

The present book being long enough as it is, one can hardly see how to have met a truly Fregean standard while covering sufficient applied mathematical ground within the covers of one or two volumes; but yet does Frege not still have a point? I believe that he does. In some Fregean sense, the book you are reading has been a book of *derivation sketches* of various merit rather than of derivations proper. I am reasonably satisfied that the book

[1]Emphasis added.

[2][112]

does what it had set out to do, but also observe that a close study of the book points out the supplementary need too for more formulaic approaches.

22.2 Temperament

The matter in question is, naturally, not one you and I are likely to settle in a few paragraphs, so for the present purpose let me turn attention to a facet of the matter which has proved significant at least to me. Though I am a practicing engineer rather than a teacher, it so happens that (besides being the father of six, which has made me a teacher of another kind) during one decade-long stretch of my career I taught, part-time, several state-university courses in electrical engineering, instructing in sum about 2000 engineering undergraduates in subjects like circuit theory, industrial electronics, continuous and discrete systems, C++ programming, and electromagnetics. My undergraduates were U.S. freshmen, sophomores and juniors mostly aged 18 to 21, so none of the teaching was very advanced; and indeed as measured in U.S. academia such occasional instructional experience as mine, sans academic research, counts for so little that I should hardly mention it here except for one point: an instructor cannot instruct so many engineering undergraduates without coming to understand somewhat of how young future engineers think and learn. When an engineering undergraduate is finding, say, Fourier's concepts hard to grasp, his engineering instructor will not extrude the topic into formulations congenial to set theory. Rather, the instructor will sketch some diagrams, assign some pencil-and-paper exercises, require the hands-on construction/testing of a suitable electromechanical apparatus, and then field questions by engaging the undergraduate's physical intuition as directly as the instructor can. Of course, professional mathematicians likewise brandish partly analogous intuitional techniques from their own instructional arsenals; but the professional carries the additional burden of preparing *his* students, by gradual stages, to join mathematics' grand investigation into foundations—or at least he carries the burden of teaching his students, in the spirit of Frege, how to deploy formal methods to preclude error. The engineering student lacks the time and, usually, the temperament for that. He has a screwdriver in his hand.

And this state of affairs is right and proper, is it not? Give the professional his due. While we engineers are off designing bridges or whatever, the professional mathematician will make it his business to be better at logic than we.

22.3 Foundations athwart intuition

The experience of writing this book has brought me to suspect that the search for the ultimate foundations of mathematics is probably futile. If man has not unearthed the ultimate foundations by now, 2500 years on from when Greek philosophy started excavating, then the coming 2500 years seem unlikely to reveal them. Some may think, "We have the computer now. We have Cantor's set theory. It is different." But I am not convinced. Not a professional mathematician nor a philosopher, I neither expect nor ask a reader to lend my conviction in such a matter much weight—nor does *this* book seriously attempt to support the conviction[3]—but as for myself, I doubt that it is in the nature of mortal human intellect to discover or grasp ultimate foundations of such a kind. Like Father Reginald, I credit St. Thomas' last report.[4]

Even so, unexpectedly, the experience of writing the book has illuminated in my sight a certain mathematical inadequacy of physical-intuitional methods. Wherever large enough a mathematical structure is built by mostly physical-intuitional methods, a fact of experience seems to emerge: the structure begins to creak.

Professional mathematicians would have told us as much.

And yet, and yet—the naïve, merely plausible extension of mathematical methods carries greater impact, and retains more power, than pure mathematics may like to admit, as in § 20.6.5 for example. Anyway, such naïve extension is more than *merely* plausible. If the objects of mathematics indeed immaterially preëxist as Plato and Frege have taught, to the extent to which the mathematician discovers rather than merely devises these objects, the ordinary French and English adjective "naïve" seems an odd one to describe the mathematician's rational[5] faculty of discovery! Observably at any rate, naïve extension yields correct mathematical results in the main, as for example in the book's applied-level development of complex exponentials and analytic continuation. Counterexamples are few, and are the fewer

[3] Try [55] and [56], rather.

[4] For reasons that have little to do with mathematics, it has been fashionable to impute such credit to fanaticism. Fashion, however, is as fleeting as it is shallow. You and I must aim deeper than that. Refer to [99, "St. Thomas Aquinas"].

[5] Scholars sometimes today lend the adjective *rational* a subtly revised meaning, a meaning incompatible with the meaning the Latin-derived adjective once bore to Leibnitz, Frege and the schoolmen (the list conspicuously excludes Plato, not because Plato would have disagreed with the schoolmen in the matter but only because Plato would as far as I know have preferred instead some variant of a Greek adjective related to our "logical"). The adjective *rational* is here used in its older, Leibnitzian-Fregean-scholastical sense.

once one has realized that contradictory definitions of *naïveté* are in play (for often all the word signifies in mathematical use is an early, presumably necessary essay at what has since grown into a better developed, more completely correct theory). No, to bend the arc of reason to follow the trail of intuition is often the right thing to do.

Substantial value subsists in the applied approach.

22.4 Convergence

One of the more debatable choices I have made during the writing of this book has been to skip explicit justification of the convergence of various sums and integrals—or, if you prefer, has been to leave the justification in most instances as an exercise. A pure mathematician would not have done so, not at any rate in the same way. I still recall an undergraduate engineering lecture, though, decades ago, during which the lean, silver-haired engineering professor—pausing the second time to justify an interchange of summing operators—rather turned to the class and confided, "Instead of justifying the interchange again, let's just *do it,* okay?" That professor had his priorities straight.

Admittedly, to train the intuition, a mathematical education probably ought at some stage to expose the student to formal, Weierstrassian tests of convergence. However, except at that stage, the repetitive justification of convergence soon grows tiresome. If the reader cannot readily tell for himself, *in a given concrete case,* whether a sum converges, is this not probably because the reader fails to understand the term being summed? If the reader indeed fails to understand, then Weierstrass can hardly help.

Though the proposition remains debatable, I believe—at least insofar as the book you are reading is an applied work—that the book's approach to convergence has been the right one.

22.5 Klein

November 2, 1895, at Göttingen, the mathematician Felix Klein (like Frege a German) masterfully summarized both sides of these matters. His lecture and its English translation having passed into the public domain, we are free to quote Klein at length as follows.

> . . . With the contemplation of nature as its starting point, and its interpretation as object, a philosophical principle, the principle

of continuity, was made fundamental; and the use of this principle characterizes the work of the great pioneers, Newton and Leibnitz, and the mathematicians of the whole of the eighteenth century—a century of discoveries in the evolution of mathematics. Gradually, however, a more critical spirit asserted itself and demanded a logical justification for the innovations made with such assurance, the establishment, as it were, of law and order after the long and victorious campaign. This was the time of Gauss and Abel, of Cauchy and Dirichlet. But this was not the end of the matter. Gauss, taking for granted the continuity of space, unhesitatingly used space intuition as a basis for his proofs; but closer investigation showed not only that many special points still needed proof, but also that space intuition had led to the too hasty assumption of the generality of certain theorems which are by no means general. Hence arose the demand for exclusively arithmetical means of proof; nothing shall be accepted as a part of the science unless its rigorous truth can be clearly demonstrated by the ordinary operations of analysis. . . . [W]here formerly a diagram served as proof, we now find continual discussions of quantities which become smaller than, or which can be taken smaller than, any given small quantity. The continuity of a variable, and what it implies, are discussed. . . .

Of course even this assigns no absolute standard of exactness; we can introduce further refinements if still stricter limitations are placed on the association of the quantities. This is exemplified . . . in the efforts to introduce symbols for the different logical processes, in order to get rid of the association of ideas, and the lack of accuracy which creeps in unnoticed, and therefore not allowed for, when ordinary language is used. . . .

Summing up all these developments in the phrase, *the arithmetizing of mathematics,* I pass on to consider the influence of the tendency here described on parts of the science outside the range of analysis proper. Thus, as you see, while voluntarily acknowledging the exceptional influence of the tendency, I do not grant that the arithmetized science is the essence of mathematics; and my remarks have therefore the two-fold character of positive approbation, and negative disapproval. For since I consider that the essential point is not the mere putting of the argument into the arithmetical form, but the more rigid logic obtained by means of this form, it seems to me desirable—and this is the positive point of my thesis—to subject the remaining divisions of mathematics to a fresh investigation based on the arithmetical foundation of analysis. On the other hand I have to point out most emphatically—and this is the negative part of my task—that it is not possible to treat mathematics exhaustively by the method of logical deduction alone. . . .

In the short time at my disposal I must content myself with presenting the most important points; I begin therefore by tracing the relation of the positive part of my thesis to the domain of geometry. The arithmetizing of mathematics began originally, as I pointed out, by ousting space intuition; the first problem that confronts us as we turn to geometry is therefore that of reconciling the results obtained by arithmetical methods with our conception of space. . . . The net result is, on the one hand, a refinement of the process of space intuition; and on the other, an advantage due to the clearer view that is hereby obtained of the analytical results considered, with the consequent elimination of the paradoxical character that is otherwise apt to attach itself to them. . . . [T]here still remains the more important question: What justification have we for regarding the totality of points in space as a number-manifoldness in which we interpolate the irrational numbers in the usual manner between the rational numbers arranged in three dimensions? We ultimately perceive that space intuition is an inexact conception, and that in order that we may subject it to mathematical treatment, we idealize it by means of the so-called axioms. . . .

Another question is this: Practical physics provides us plentifully with experimental results, which we unconsciously generalize and adopt as theorems about the idealized objects. . . . [T]he theorem that every finite elastic body is capable of an infinite series of harmonic oscillations [belongs to this category]. . . . [Is such a theorem], taken in the abstract, [an] exact mathematical [theorem], or how must [it] be limited and defined in order that [it] may become so? . . . You see here what is the precise object of . . . renewed investigations; not any new physical insight, but abstract mathematical argument in itself, on account of the clearness and precision which will thereby be added to our view of experimental facts. If I may use an expression of Jacobi's in a somewhat modified sense, it is merely a question of intellectual integrity, "die Ehre des menschlichen Geistes."

After expressing myself thus it is not easy, without running counter to the foregoing conclusions, to secure to intuition her due share in our science;[6] and yet it is exactly on this antithesis that the point of my present statements depends. I am now thinking not so much of the cultivated intuition just discussed, which has been developed under the influence of logical deduction and might almost be called a form of memory; but rather of the naïve intuition, largely a natural gift, which is unconsciously increased by minute study of one branch or other of the science. The word *intuition* is perhaps not well chosen; I mean it to include that instinctive feeling for the proportion of the moving parts with which the engineer criticises the distribution of power in any piece of mechanism he has constructed; and even the indefinite conviction the practiced calculator possesses as to the convergence of any infinite process that lies before him. I maintain that mathematical intuition—so understood—is always far in advance of logical reasoning and covers a wider field.

I might now introduce an historical excursus, showing that in the development of most of the branches of our science, intuition was the starting point, while logical treatment followed. . . . The question in all such cases, to use the language of analysis, is one of interpolation, in which less stress is laid on exactness in particular details than on a consideration of the general conditions. . . . Logical investigation is not in place until intuition has completed the task of idealization. . . .

I must add a few words on mathematics from the point of view of pedagogy. We observe in Germany at the present day a very remarkable condition of affairs in this respect; two opposing currents run side by side without affecting one another appreciably. Among instructors in our Gymnasia [that is, roughly as understood in North American terms, in Germany's elite, preparatory high schools] the need of mathematical instruction based on intuitive methods has now been so strongly and universally emphasized that one is compelled to enter a protest, and vigorously insist on the necessity for strict logical treatment. . . . Among

[6] Klein characterizes mathematics as a "science" so often that, insofar as my book is quoting Klein with approbation, I should note that I have never been persuaded that *science* is the right word for it. This is a minor quibble, and my witness may not weigh much in comparison with that of the eminent Klein, but I would nevertheless prefer rather to regard mathematics as a branch of *philosophy*.

the university professors of our subject exactly the reverse is the case; intuition is frequently not only undervalued, but as much as possible ignored. This is doubtless a consequence of the intrinsic importance of the arithmetizing tendency in modern mathematics. But the result reaches far beyond the mark. It is high time to assert openly once for all that this implies, not only a false pedagogy, but also a distorted view of the science. I ... have always discouraged the laying-down of general rules for higher mathematical teaching, but this shall not prevent me from saying that two classes at least of mathematical lectures must be based on intuition; the elementary lectures which actually introduce the beginner to higher mathematics—for the scholar must naturally follow the same course of development on a smaller scale, that the science itself has taken on a larger—and the lectures which are intended for those whose work is largely done by intuitive methods, namely, natural scientists and engineers. Through this one-sided adherence to logical form we have lost among these classes of men much of the prestige properly belonging to mathematics, and it is a pressing and urgent duty to regain this prestige by judicious treatment.

To return to theoretical considerations, the general views which I uphold in regard to the present problems of mathematical science need scarcely be specially formulated. While I desire in every case the fullest logical working out of the material, yet I demand at the same time an intuitive grasp and investigation of the subject from all sides. Mathematical developments originating in intuition must not be considered actual constituents of the science till they have been brought into a strictly logical form. Conversely, the mere abstract statements of logical relations cannot satisfy us until the extent of their application to every branch of intuition is vividly set forth, and we recognize the manifold connections of the logical scheme, depending on the branch which we have chosen, to the other divisions of our knowledge. The science of mathematics may be compared to a tree thrusting its roots deeper into the earth and freely spreading out its shady branches into the air. Are we to consider the roots or the branches as the essential part? Botanists tell us that the question is badly framed, and that the life of the organism depends on the mutual interaction of its different parts. [98]

Klein's standard is not a standard against which the book whose end you have reached—the book being, after all, a book of derivations of *applied* mathematics—should precisely seek to measure itself; yet more than a century after Klein spoke, I can still think of no more fitting way to end the book than with Klein's robust reflections. During quiet moments, when the applied mathematician is not out throwing bridges across chasms and such, he may well ponder that which Klein has taught.

THB

Appendices

Appendix A

Hexadecimal and other notational matters

Convention is a practical necessity in mathematical notation. Consider

$$\rotatebox{90}{2}\,;\,|\,;\,\rotatebox{-90}{5}\,\triangleright\,\cdots\,\Box$$

which (say) means "two plus two equals four," as long as one grasps that ";" means two, "$\Box$" means four, "$\rotatebox{90}{2}\,|\,\rotatebox{-90}{5}\,\triangleright$" indicates addition, and so on. No one can actually read such hieroglyphics, though, whereas

$$2 + 2 = 4$$

is immediately readable without further explication, however arbitrary conventional symbols like "2," "4," "+," and so on might seem. And how arbitrary are the conventional symbols, really? Well, individually, the conventional symbols are arbitrary enough, yet the received body of conventional mathematical notation is not *merely* arbitrary. The notation is worn to fit like an old boot, rather, centuries of mathematical practice being the foot which has broken the boot in. Today, if one wishes to write mathematical ideas to others—and indeed, very likely, if one wishes to write mathematical ideas even to oneself—then one will probably follow convention in one's choice of notation.

Sometimes however the conventional notation can embody the wrong idea. Sometimes the notation inadvertently suggests a thought, or lends a perspective, the writer writing it never meant. Sometimes the notation is merely awkward, as in Girolamo Cardan's 1539 letter to Tartaglia:

> [T]he cube of one-third of the coefficient of the unknown is greater in value than the square of one-half of the number. [122]

Cardan means,

$$\left(\frac{a}{3}\right)^3 > \left(\frac{x}{2}\right)^2.$$

Fortunately, convention has since 1539 condensed such notation, as you see; so Cardan's trouble has little troubled this book.

A.1 The symbol 2π

What has troubled this book, slightly, is the familiar

$$\pi \approx 3.1416.$$

The notation is familiar because it is conventional, and it is conventional because, well, that is just how people write mathematics. Nevertheless, for this book's purpose, the expression as written—though conventional and indisputably accurate—inadvertently introduces two questionable ideas the book's writer never intended to entertain:

- it inadvertently introduces the idea that π as such were a quantity of special interest (it isn't);
- it inadvertently introduces the idea that the tenth, hundredth, thousandth and ten thousandth parts of π deserved the reader's notice (they don't).

Whether such ideas might merit attention in another book is not a question the appendix you are reading will consider. The question rather is whether the ideas have come *here,* unbidden, as uninvited guests.

The book prefers the notation

$$2\pi \approx \text{0x6.487F}.$$

Though less conventional, this notation dismisses the guests.

Taken as a single symbol 2π seems preferable to π (for π after all represents only half a circle), but 2π taken as a single symbol remains, maybe, visually still a bit awkward. One wants instead to introduce some new symbol[1] $\xi = 2\pi$, $\tau = 2\pi$ or $\varpi\!\!\pi = 2\pi$ (the last being slightly preferred by this writer). However, inasmuch as the book has already stretched to reach the more substantive § A.2, next, caution here prevails and the style 2π is retained.

[1]See among others [124][12][110].

A.2 Hexadecimal numerals

Treating 2π sometimes as a single symbol is a small step, unlikely to trouble readers much. A bolder step is to adopt from the computer-science and computer-engineering literature the important notational improvement of the hexadecimal numeral. No incremental step is possible here; either we leap the ditch or we remain on the wrong side. In this book, we choose to leap.

Traditional decimal notation seems unobjectionable for measured quantities like 63.7 miles, \$1.32 million or 9.81 m/s^2, but its iterative tenfold structure meets little or no aesthetic support in mathematical theory. Consider for instance the decimal numeral 127, whose number suggests a significant idea to the computer scientist or computer engineer but whose decimal notation does nothing to convey the notion of the largest signed integer storable in a byte. Better is the base-sixteen hexadecimal notation 0x7F, which clearly expresses the idea of $2^7 - 1$. To the reader who is not a computer scientist or computer engineer, the aesthetical advantage may not seem immediately clear from the one example, but consider the decimal numeral 2,147,483,647, which represents the largest signed integer storable in a standard thirty-two bit word. In hexadecimal notation, this is 0x7FFF FFFF, or in other words $2^{\mathrm{0x1F}} - 1$. The question is: which notation more clearly captures the idea?

By contrast, decimal notation like 499,999 does not really convey any interesting *mathematical* idea at all, except with regard to a special focus on tens—a focus which is of immense practical use but which otherwise tells one very little about numbers, as numbers. Indeed, one might go so far as to say that the notation 499,999 were misleading, insofar as it attaches mathematically false interest to the idea it represents. (The hexadecimal representation 0x7A11F = 499,999 by contrast suggests to the eye at once the arguably, relatively insignificant character of the number in question. So, which is the more interesting quantity, all things considered? A 2,147,483,647 or a 499,999?)

Now, one does not wish to sell the hexadecimal numeral too hard. Decimal numerals are fine: the author uses them as often, and likes them as well, as almost anyone does. Familiar idiosyncrasy has value, after all. Nevertheless, the author had a choice when writing this book, and for *this* book the hexadecimal numeral seemed the proper, conceptually elegant choice—proper and conceptually elegant enough indeed to risk deviating this far from convention—so that is the numeral he chose.

To readers unfamiliar with the hexadecimal notation, to explain very briefly: hexadecimal represents numbers not in tens but rather in sixteens. The rightmost place in a hexadecimal numeral represents ones; the next place leftward, sixteens; the next place leftward, sixteens squared; the next, sixteens cubed, and so on. For instance, the hexadecimal numeral 0x1357 means "seven, plus five times sixteen, plus thrice sixteen times sixteen, plus once sixteen times sixteen times sixteen" (all of which totals to 4951 in decimal). In hexadecimal, the sixteen symbols 0123456789ABCDEF respectively represent the numbers zero through fifteen, with sixteen being written 0x10.

All this raises the sensible question: why sixteen?[2] The answer is that sixteen is 2^4, so hexadecimal (base sixteen) is found to offer a convenient shorthand for binary (base two, the fundamental, smallest possible base). Each of the sixteen hexadecimal digits represents a unique sequence of exactly four bits (binary digits). Binary is inherently theoretically interesting, but direct binary notation is unwieldy (the hexadecimal numeral 0x1357 is binary 0001 0011 0101 0111), so hexadecimal is written in proxy.

Admittedly, the hexadecimal "0x" notation is bulky and overloads the letters A through F (letters which otherwise conventionally often represent matrices or indeterminate coefficients). However, the greater trouble with the hexadecimal notation is not in the notation itself but rather in the unfamiliarity with it. The reason it is unfamiliar is that it is not often encountered outside the computer-science and computer-engineering literature, but it is not encountered because it is not used, and it is not used because it is not familiar, and so on in a cycle. It seems to this writer, on aesthetic grounds, that this particular cycle is worth breaking, so the book you are reading employs the hexadecimal system for integers larger than 9. If you have never yet used the hexadecimal system, it is worth your while to learn it. For the sake of conceptual elegance, at the risk of transgressing entrenched convention, this book employs hexadecimal throughout.

The book occasionally omits the cumbersome hexadecimal prefix "0x," as for example when it arrays hexadecimal numerals in matrices (as in § 12.3.1 where A is ten but, unfortunately potentially confusingly, A, set

[2] An alternative [104, book 6, no. 83] advocated by some nineteenth-century writers was twelve. (Laplace, cited, was not indeed one of the advocates, or at any rate was not a strong advocate; however, his context appears to have lain in the promotion of base twelve by contemporaries.) In base twelve, one quarter, one third and one half are respectively written 0.3, 0.4 and 0.6. Also, the hour angles (§ 3.6) come in neat increments of $(0.06)(2\pi)$ in base twelve, so there are some real advantages to that base. Hexadecimal, however, besides having momentum from the computer-science and computer-engineering literature, is preferred for its straightforward proxy of binary.

in italics, is a matrix; and as in Fig. 4.2).

The book seldom mentions numbers with physical units of measure attached but, when it does, it expresses those in decimal rather than hexadecimal notation—for example, $v_{\text{sound}} = 331$ m/s rather than $v_{\text{sound}} =$ 0x14B m/s.

A.3 Avoiding notational clutter

Good applied mathematical notation is not cluttered. Good notation does not necessarily include every possible limit, qualification, superscript and subscript. For example, the sum

$$S = \sum_{i=1}^{M} \sum_{j=1}^{N} a_{ij}^2$$

might be written less thoroughly but more readably as

$$S = \sum_{i,j} a_{ij}^2$$

if the meaning of the latter were clear from the context.

When to omit subscripts and such is naturally a matter of style and subjective judgment, but in practice such judgment is often not hard to render. The balance is between showing few enough symbols that the interesting parts of an equation are not obscured visually in a tangle and a haze of redundant little letters, strokes and squiggles, on the one hand; and on the other hand showing enough detail that the reader who opens the book directly to the page has a fair chance to understand what is printed there without studying the whole book carefully up to that point. Where appropriate, this book often condenses notation and omits redundant symbols.

Appendix B

The Greek alphabet

Mathematical experience finds the Roman alphabet to lack sufficient symbols to write higher mathematics fluently. Though not completely solving the problem, the addition of the Greek alphabet helps. See Table B.1.

When first seen in English-language mathematical writing, the Greek letters can seem to take on a wise, mysterious aura. Nevertheless, the Greek letters are just letters. We use them not because we want to be wise and mysterious[1] but rather because we simply do not have enough Roman letters. An equation like

$$\alpha^2 + \beta^2 = \gamma^2$$

says no more than does an equation like

$$a^2 + b^2 = c^2,$$

after all. The letters are just different (though naturally one prefers to use the letters one's audience expects when one can).

Applied as well as professional mathematicians tend to use Roman and Greek letters in certain long-established conventional sets: $abcd$; fgh; $ijk\ell$; mn; pqr; st; uvw; xyz. For the Greek: $\alpha\beta\gamma$; $\delta\epsilon$; $\kappa\lambda\mu\nu\xi$; $\rho\sigma\tau$; $\phi\chi\psi\omega$. Greek

[1] Well, you can use them to be wise and mysterious if you want to. It's kind of fun, actually, when you're dealing with someone who doesn't understand math—if what you want is for him to go away and leave you alone. Otherwise, we tend to use Roman and Greek letters in various conventional ways: Greek minuscules (lower-case letters) for angles; Roman capitals for matrices; e for the natural logarithmic base; f and g for unspecified functions; i, j, k, m, n, M and N for integers; P and Q for logical propositions and metasyntactic elements; t, T and τ for time; d, δ and Δ for change; A, B and C for indeterminate coefficients; etc.

Table B.1: The Roman and Greek alphabets.

ROMAN

Aa	Aa	Gg	Gg	Mm	Mm	Tt	Tt
Bb	Bb	Hh	Hh	Nn	Nn	Uu	Uu
Cc	Cc	Ii	Ii	Oo	Oo	Vv	Vv
Dd	Dd	Jj	Jj	Pp	Pp	Ww	Ww
Ee	Ee	Kk	Kk	Qq	Qq	Xx	Xx
Ff	Ff	$L\ell$	Ll	Rr	Rr	Yy	Yy
				Ss	Ss	Zz	Zz

GREEK

$\mathrm{A}\alpha$	alpha	$\mathrm{H}\eta$	eta	$\mathrm{N}\nu$	nu	$\mathrm{T}\tau$	tau
$\mathrm{B}\beta$	beta	$\Theta\theta$	theta	$\Xi\xi$	xi	$\Upsilon\upsilon$	upsilon
$\Gamma\gamma$	gamma	$\mathrm{I}\iota$	iota	$\mathrm{O}o$	omicron	$\Phi\phi$	phi
$\Delta\delta$	delta	$\mathrm{K}\kappa$	kappa	$\Pi\pi$	pi	$\mathrm{X}\chi$	chi
$\mathrm{E}\epsilon$	epsilon	$\Lambda\lambda$	lambda	$\mathrm{P}\rho$	rho	$\Psi\psi$	psi
$\mathrm{Z}\zeta$	zeta	$\mathrm{M}\mu$	mu	$\Sigma\sigma$	sigma	$\Omega\omega$	omega

letters are frequently paired with their Roman congeners as appropriate: $a\alpha$; $b\beta$; $cg\gamma$; $d\delta$; $e\epsilon$; $f\phi$; $k\kappa$; $\ell\lambda$; $m\mu$; $n\nu$; $p\pi$; $r\rho$; $s\sigma$; $t\tau$; $hx\chi$; $z\zeta$.[2]

[2] The capital pair $Y\Upsilon$ is occasionally seen but is awkward both because the Greek minuscule υ is visually almost indistinguishable from the unrelated (or distantly related) Roman minuscule v; and because the ancient Romans regarded the letter Y not as a congener but as the Greek letter itself, seldom used but to spell Greek words in the Roman alphabet. To use Y and Υ as separate symbols is to display an indifference to, easily misinterpreted as an ignorance of, the Graeco-Roman sense of the thing—which is silly, arguably, if you think about it, since no one objects when you differentiate j from i, or u and w from v—but, anyway, one is probably the wiser to tend to limit the mathematical use of the symbol Υ to the very few instances in which established convention decrees it. (In English particularly, there is also an old typographical ambiguity between Y and a Germanic, non-Roman letter Þ, lower-case þ, named "thorn," which has practically vanished from English today, to the point that the typeface in which you are reading these words has required the loading of a special annex merely to render a glyph for it—but which sufficiently literate writers are still expected to recognize on sight. The thorn does not look much like a Y or y in the present typeface but in some other typefaces it does—which incidentally is why the medieval English article "the," "þe," was in the 20th-century United States sometimes printed, and thence in jest pronounced, as "ye"—as in "Ye Olde English Shoppe," painted "Ye Olde English Shoppe" on a sign above a tourist trap of a gift shop your author remembers visiting as a child as late as the 1970s; which of course has little properly to do with the letter thorn but there it is. Such obscure pop-cultural

Mathematicians usually avoid letters like the Greek capital H (eta), which looks just like the Roman capital H, even though H (eta) is an entirely proper member of the Greek alphabet. The Greek minuscule υ (upsilon) is avoided for like reason, for mathematical symbols are useful only insofar as one can visually tell them apart. Interestingly, however, the Greek minuscules ν (nu) and ω (omega) are often used in applied mathematics, so one needs to learn to distinguish those ones from the Roman v and w.

references, though oblique, afford yet one more reason to tend to avoid Υ when you can, a Greek capital that makes you look indifferent or ignorant when you use it wrong and ironic or pretentious when you use it right. You can't win.)

The history of the alphabets is extremely interesting. Unfortunately, a footnote in an appendix to a book on derivations of applied mathematics is probably not the right place for an essay on the topic, so we'll let the matter rest there.

Appendix C

A bare sketch of the pure theory of the complex variable

At least three of the various disciplines of pure mathematics stand out for their pedagogical intricacy and the theoretical depth of their core results. The first of the three is number theory which, except for the simple results of § 6.1, scientists and engineers tend to get by largely without. The second is matrix theory (chapters 11 through 14), a bruiser of a discipline the applied mathematician of the computer age—try though he might—can hardly escape. The third is the pure theory of the complex variable.

The introduction's § 1.4 admires the beauty of the pure theory of the complex variable even while admitting that that theory's "arc regrettably takes off too late and flies too far from applications for such a book as this." To develop the pure theory properly is a worthy book-length endeavor of its own requiring moderately advanced preparation on its reader's part which, however, the reader who has reached the end of the present book's chapter 9 (or even of its section § 8.9) possesses. If the writer doubts the strictly applied *necessity* of the pure theory, still, he does not doubt its health to one's overall mathematical formation. It provides another way to think about complex numbers. Scientists and engineers with advanced credentials occasionally expect one to be acquainted with it for technical-social reasons, regardless of its practical use. Besides, the pure theory is interesting. This alone recommends some attention to it.[1]

[1]G. E. Shilov more thoroughly covers the pure theory in his [147] and [148]. Though in the sense of the Debian Free Software Guidelines nonfree (§ 1.5), Shilov's coverage is

The pivotal result of pure complex-variable theory is the Taylor series by Cauchy's impressed residue theorem. If we will let these few pages of appendix replace an entire book on the pure theory, then Cauchy's and Taylor's are the results we will sketch. The bibliography lists presentations far more complete.

Cauchy's impressed residue theorem[2] is that

$$f(z) = \frac{1}{i2\pi} \oint \frac{f(w)}{w-z}\, dw \tag{C.1}$$

if z lies within the closed complex contour about which the integral is taken and if $f(z)$ is everywhere analytic (§ 8.4) within and along the contour. More than one proof of the theorem is known, depending on the assumptions from which the mathematician prefers to start, but this writer is partial to an instructively clever proof he has learned from D. N. Arnold[3] which goes as follows. Consider the function

$$g(z,t) \equiv \frac{1}{i2\pi} \oint \frac{f[z+(t)(w-z)]}{w-z}\, dw,$$

whose derivative with respect to the parameter t is[4]

$$\frac{\partial g}{\partial t} = \frac{1}{i2\pi} \oint f'[z+(t)(w-z)]\, dw.$$

We notice that this is

$$\begin{aligned} \frac{\partial g}{\partial t} &= \frac{1}{i2\pi} \oint \frac{\partial}{\partial w} \left\{ \frac{f[z+(t)(w-z)]}{t} \right\} dw \\ &= \frac{1}{i2\pi} \left\{ \frac{f[z+(t)(w-z)]}{t} \right\}_{w=a}^{b}, \end{aligned}$$

mentioned here because its style is comparatively accessible to the scientist or engineer.

[2]This is not a standard name. Though they name various associated results after Cauchy in one way or another, neither [79] nor [7] seems to name this particular result, though both do feature it. Since (C.1) impresses a pole and thus also a residue on a function $f(z)$ which in the domain of interest lacks them, the name *Cauchy's impressed residue theorem* ought to serve this appendix's purpose ably enough.

[3][7, § III]

[4]The book does not often employ Newton's notation $f'(\cdot) \equiv [(d/d\zeta)f(\zeta)]_{\zeta=(\cdot)}$ of § 4.4 but the notation is handy here because it evades the awkward circumlocution of changing $\zeta \leftarrow z$ in (C.1) and then writing,

$$\frac{\partial g}{\partial t} = \frac{1}{i2\pi} \oint \frac{[(d/d\zeta)f(\zeta)]_{\zeta=z+(t)(w-z)}}{w-z}\, dw.$$

where a and b respectively represent the contour integration's beginning and ending points. But this integration ends where it begins and its integrand (lacking a w in the denominator) is analytic within and along the contour, so $a = b$ and the factor $\{\cdot\}_{w=a}^{b}$ in braces vanishes, whereupon

$$\frac{\partial g}{\partial t} = 0,$$

meaning that $g(z, t)$ does not vary with t. Observing per (8.26) that

$$\frac{1}{i2\pi} \oint \frac{dw}{w - z} = 1,$$

we have that

$$f(z) = \frac{f(z)}{i2\pi} \oint \frac{dw}{w - z} = g(z, 0) = g(z, 1) = \frac{1}{i2\pi} \oint \frac{f(w)}{w - z}\, dw$$

as was to be proved. (There remains a basic question as to whether the paragraph's integration is even valid. Logically, it ought to be valid, since $f[z]$ being analytic is infinitely differentiable,[5] but when the integration is used as the sole theoretical support for the entire calculus of the complex variable, well, it seems an awfully slender reed to carry so heavy a load. Admittedly, maybe this is only a psychological problem, but a professional mathematician will devote many pages to preparatory theoretical constructs before even attempting the integral, the result of which lofty effort is not in the earthier spirit of applied mathematics. On the other hand, now that the reader has followed the book along its low road and the high integration is given only in reserve, now that the integration reaches a conclusion already believed and, once there, is asked to carry the far lighter load of this appendix only, the applied reader may feel easier about trusting it.)

One could follow Arnold hence toward the proof of the theorem of one Goursat and further toward various other interesting results, a path of study the writer recommends to sufficiently interested readers: see [7]. Being in a tremendous hurry ourselves, however, we will leave Arnold and follow F. B. Hildebrand[6] directly toward the Taylor series. Positing some expansion point z_o and then expanding (C.1) geometrically per (2.36) about it, we

---

[5] The professionals minimalistically actually require only that the function be once differentiable under certain conditions, from which they prove infinite differentiability, but this is a fine point which will not concern us here.

[6] [79, § 10.7]

have that

$$
\begin{aligned}
f(z) &= \frac{1}{i2\pi} \oint \frac{f(w)}{(w-z_o)-(z-z_o)}\,dw \\
&= \frac{1}{i2\pi} \oint \frac{f(w)}{(w-z_o)[1-(z-z_o)/(w-z_o)]}\,dw \\
&= \frac{1}{i2\pi} \oint \frac{f(w)}{w-z_o} \sum_{k=0}^{\infty} \left[\frac{z-z_o}{w-z_o}\right]^k dw \\
&= \sum_{k=0}^{\infty} \left\{ \left[\frac{1}{i2\pi} \oint \frac{f(w)}{(w-z_o)^{k+1}}\,dw \right] (z-z_o)^k \right\},
\end{aligned}
$$

which, being the power series

$$
\begin{aligned}
f(z) &= \sum_{k=0}^{\infty} (a_k)(z-z_o)^k, \\
a_k &\equiv \frac{1}{i2\pi} \oint \frac{f(w)}{(w-z_o)^{k+1}}\,dw,
\end{aligned}
\tag{C.2}
$$

by definition constitutes the Taylor series (8.19) for $f(z)$ about $z = z_o$, assuming naturally that $|z - z_o| < |w - z_o|$ for all w along the contour so that the geometric expansion above will converge.

The important theoretical implication of (C.2) is that *every function has a Taylor series about any point across whose immediate neighborhood the function is analytic.* There evidently is no such thing as an analytic function without a Taylor series—a fact we already knew if we have read and believed chapter 8, but some readers may find it more convincing this way.

Comparing (C.2) against (8.19), incidentally, we have also that

$$
\left.\frac{d^k f}{dz^k}\right|_{z=z_o} = \frac{k!}{i2\pi} \oint \frac{f(w)}{(w-z_o)^{k+1}}\,dw, \tag{C.3}
$$

which is an alternate way to write (8.31).

Close inspection of the reasoning by which we have reached (C.2) reveals, quite by the way, at least one additional result which in itself tends to vindicate the pure theory's technique. It is this: that *a Taylor series remains everywhere valid out to the distance of the nearest nonanalytic point.* The proposition is explained and proved as follows. For the aforementioned contour of integration nothing prevents one from choosing a circle, centered in the Argand plane on the expansion point $z = z_o$, the circle's radius just as large as it can be while still excluding all nonanalytic points. The requirement that $|z - z_o| < |w - z_o|$ for all w along the contour evidently is met for all z inside such a circle, which means that the Taylor series (C.2) converges for all z inside the circle, which—precisely because we have stipulated that the circle be the largest possible centered on the expansion point—implies and thus proves the proposition in question. As an example of the proposition's use, consider the Taylor series Table 8.1 gives for $-\ln(1 - z)$, whose nearest nonanalytic point at $z = 1$ lies at unit distance from the series' expansion point $z = 0$: according to the result of this paragraph, the series in question remains valid over the Argand circle out to unit distance, $|z| < 1$.

Appendix D

The irrationality of 2π

That $\sqrt{2}$, $\sqrt{3}$ and so on are irrational has been proved in § 6.1.4. That 2π is irrational has however not been proved in the body of the book. How important it is to applications to prove that 2π is irrational can be questioned, but the proposition that 2π is irrational is at least intriguing. Unfortunately, every proof the writer has encountered wants either technique that strays far from the applicationary domain or an esoteric degree of cleverness,[1] or both.

An esoterically clever but logical, instructive, reasonably brief proof that prerequires an understanding of the book's contents up through and including § 8.12 has been conjured by Ivan Niven [120][115] as follows.[2] Let

$$\pi = \frac{a}{b}, \qquad (a,b) \in \mathbb{Z},\ a > 0,\ b > 0, \tag{D.1}$$

[1]But do we not admire cleverness? Answer: we might, but being applied mathematicians rather than pure we might have admired a better-motivated, more workmanlike technique more. Still, even an applied mathematician can profit from acquaintance with the specific technique this appendix reviews; and, admittedly, once one has reached the end of the appendix and seen how the technique works, the motivation to have used the technique here waxes somewhat less opaque.

Anyway, as the narrative confesses, the writer knows no workmanlike technique to prove the irrationality of 2π. If he did know one, he probably would have reported that one, instead.

[2]This appendix closely follows Niven in [120]. Niven's lucid, economical original fits on a single page, but this appendix expresses Niven's proof in a somewhat less economical manner that better suits the mode of the present book.

be a rational number. Defining

$$f(x) \equiv \frac{(x)^n(a-bx)^n}{n!}, \qquad n \in \mathbb{Z},\ n \ge 0, \tag{D.2}$$

the n being an arbitrarily large integer, one observes that

$$\begin{aligned}
f(x) &= \frac{1}{n!}\sum_{j=0}^{n} c_j x^{n+j}, && c_j \in \mathbb{Z}, \\
\frac{d^k f}{dx^k} &= \frac{1}{n!}\sum_{j=0}^{n} \frac{(n+j)!}{(n+j-k)!} c_j x^{n+j-k} && \text{for } 0 \le k < n, \\
\frac{d^k f}{dx^k} &= \frac{1}{n!}\sum_{j=k-n}^{n} \frac{(n+j)!}{(n+j-k)!} c_j x^{n+j-k} && \text{for } n \le k \le 2n, \\
\frac{d^k f}{dx^k} &= 0 && \text{for } 2n < k,
\end{aligned} \tag{D.3}$$

in which the several c_j are integers because a and b are integers. Evaluating at $x = 0$,

$$\begin{aligned}
\left.\frac{d^k f}{dx^k}\right|_{x=0} &= 0 && \text{for } 0 \le k < n, \\
\left.\frac{d^k f}{dx^k}\right|_{x=0} &= \frac{k!}{n!} c_{k-n} && \text{for } n \le k \le 2n, \\
\left.\frac{d^k f}{dx^k}\right|_{x=0} &= 0 && \text{for } 2n < k,
\end{aligned} \tag{D.4}$$

for all of which

$$\left.\frac{d^k f}{dx^k}\right|_{x=0} \in \mathbb{Z} \tag{D.5}$$

because (as just noted) the several c_{k-n} are integers and because, where the derivative is nonzero, $k \ge n$.

Having assumed in (D.1) that $\pi = a/b$ were rational, one can write (D.2) as

$$f(x) = b^n \frac{(x)^n(\pi - x)^n}{n!}, \tag{D.6}$$

which makes plain that $f(\pi - u) = f(u)$ and thus that

$$\left.\frac{d^k f}{dx^k}\right|_{x=\pi-u} = (-)^k \left.\frac{d^k f}{dx^k}\right|_{x=u}.$$

Evaluating this at $u = 0$ and recalling (D.5),

$$\left.\frac{d^k f}{dx^k}\right|_{x=\pi} = (-)^k \left.\frac{d^k f}{dx^k}\right|_{x=0} \in \mathbb{Z}. \tag{D.7}$$

Defining

$$F(x) \equiv \sum_{j=0}^{n} (-)^j \frac{d^{2j} f}{dx^{2j}}, \tag{D.8}$$

and heeding (D.3) that $d^k f/dx^k = 0$ for all $k > 2n$, we have that

$$\frac{d}{dx}\left[\frac{dF}{dx}\sin x - F(x)\cos x\right] = \frac{d^2 F}{dx^2}\sin x + F(x)\sin x = f(x)\sin x,$$

whereby

$$\int_0^\pi f(x)\sin x\,dx = \left[\frac{dF}{dx}\sin x - F(x)\cos x\right]_0^\pi.$$

That is,

$$\int_0^\pi f(x)\sin x\,dx = F(\pi) + F(0). \tag{D.9}$$

From this stage, after noticing that (D.2) imposes a ceiling on $f(x)$ within the interior interval, we may quote Niven's conclusion as follows.[3]

> Now $F(\pi) + F(0)$ is an *integer,* since $d^k f/dx^k|_{x=\pi}$ and $d^k f/dx^k|_{x=0}$ are integers. But for $0 < x < \pi$,
>
> $$0 < f(x)\sin x < \frac{\pi^n a^n}{n!},$$
>
> so that the integral in (D.9) is *positive, but arbitrarily small* for n sufficiently large. Thus (D.9) is false, and so is our assumption that π is rational.

And 2π is no more rational than π is.

[3]This appendix's rendition of the quotation prints $d^k f/dx^k$ in place of Niven's notation for the derivative and, of course, replaces the equation number Niven uses by a number that resolves within this appendix. The quotation's italics are Niven's own.

Appendix E

Manuscript history

The book in its present form is based on various unpublished drafts and notes of mine, plus a few of my wife Kristie's (née Hancock), going back to 1983 when I was fifteen years of age. What prompted the contest I can no longer remember, but the notes began one day when I challenged a high-school classmate to prove the quadratic formula. The classmate responded that he did not need to prove the quadratic formula because the proof was in the class' math textbook and then counterchallenged me to prove the Pythagorean theorem. Admittedly obnoxious (I was fifteen, after all) but not to be outdone, I whipped out a pencil and paper on the spot and started working, but I found that I could not prove the theorem that day.

The next day I did find a proof in the school's library,[1] writing it down, adding to it the proof of the quadratic formula plus a rather inefficient proof of my own invention to the law of cosines. Soon thereafter the school's chemistry instructor happened to mention that the angle between the tetrahedrally arranged four carbon-hydrogen bonds in a methane molecule was 109°, so from a symmetry argument I proved that result to myself, too, adding it to my little collection of proofs. That is how it started.[2]

The book actually has earlier roots than these. In 1979, when I was twelve years old, my father bought our family's first eight-bit computer. The computer's built-in *BASIC* programming-language interpreter exposed

[1]Better proofs are found in § 2.9.4 and the introduction to chapter 1.

[2]Fellow gear-heads who lived through that era at about the same age might want to date me against the disappearance of the slide rule. Answer: in my country, or at least at my high school, I was three years too young to use a slide rule. The kids born in 1964 learned the slide rule; those born in 1965 did not. I wasn't born till 1967, so for better or for worse I always had a pocket calculator in high school. My family had an eight-bit computer at home, too, as we shall see.

functions for calculating sines and cosines of angles. The interpreter's manual included a diagram much like Fig. 3.1 showing what sines and cosines were, but it never explained how the computer went about calculating such quantities. This bothered me at the time. Many hours with a pencil I spent trying to figure it out, yet the computer's trigonometric functions remained mysterious to me. When later in high school I learned of the use of the Taylor series to calculate trigonometrics, into my growing collection of proofs the series went.

Five years after the Pythagorean incident I was serving the U.S. Army as an enlisted troop in the former West Germany. Although those were the last days of the Cold War, there was no shooting war at the time, so the duty was peacetime duty. My duty was in military signal intelligence, frequently in the middle of the German night when there often wasn't much to do. The platoon sergeant wisely condoned neither novels nor playing cards on duty, but he did let the troops read the newspaper after midnight when things were quiet enough. Sometimes I used the time to study my German—the platoon sergeant allowed this, too—but I owned a copy of Richard P. Feynman's *Lectures on Physics* [57] which sometimes I would read instead.

Late one night the battalion commander, a lieutenant colonel and West Point graduate, inspected my platoon's duty post by surprise. A lieutenant colonel was a highly uncommon apparition at that hour of a quiet night, so when that old man appeared suddenly with the sergeant major, the company commander and the first sergeant in tow—the last two just routed from their sleep, perhaps—surprise indeed it was. The colonel may possibly have caught some of my unlucky fellows playing cards that night—I am not sure—but me, he caught with my boots unpolished, reading the *Lectures.*

I snapped to attention. The colonel took a long look at my boots without saying anything, as stormclouds gathered on the first sergeant's brow at his left shoulder, and then asked me what I had been reading.

"Feynman's *Lectures on Physics,* sir."

"Why?"

"I am going to attend the university when my three-year enlistment is up, sir."

"I see." Maybe the old man was thinking that I would do better as a scientist than as a soldier? Maybe he was remembering when he had had to read some of the *Lectures* himself at West Point. Or maybe it was just the singularity of the sight in the man's eyes, as though he were a medieval knight at bivouac who had caught one of the peasant levies, thought to be illiterate, reading Cicero in the original Latin. The truth of this, we shall never know. What the old man actually said was, "Good work, son. Keep it up."

The stormclouds dissipated from the first sergeant's face. No one ever said anything to me about my boots (in fact as far as I remember, the first sergeant—who saw me seldom in any case—never spoke to me again). The platoon sergeant thereafter explicitly permitted me to read the *Lectures* on duty after midnight on nights when there was nothing else to do, so in the last several months of my military service I did read a number of them. It is fair to say that I also kept my boots better polished.

In Volume I, Chapter 6, of the *Lectures* there is a lovely introduction to probability theory. It discusses the classic problem of the "random walk" in some detail and then states without proof that the generalization of the random walk leads to the Gaussian distribution (§ 20.5),

$$\Omega(x) = \frac{\exp(-x^2/2\sigma^2)}{\sigma\sqrt{2\pi}}.$$

For the derivation of this remarkable theorem, I scanned the book in vain. One had no Internet access in those days, but besides a well-equipped gym the Army post also had a tiny library, and in one yellowed volume in the library—who knows how such a book got there?—I did find a derivation of the $1/\sigma\sqrt{2\pi}$ factor.[3] The exponential factor, the volume did not derive. Several days later, I chanced to find myself in Munich with an hour or two to spare, which I spent in the university library seeking the missing part of the proof, but lack of time and unfamiliarity with such a German site defeated me. Back at the Army post, I had to sweat the proof out on my own over the ensuing weeks. Nevertheless, eventually I did obtain a proof which made sense to me. Writing the proof down carefully, I pulled the old high-school math notes out of my military footlocker (for some reason I had kept the notes and even brought them to Germany), dusted them off, and added to them the new Gaussian proof.

That is how it has gone. To the old notes, I have added new proofs from time to time, and though I have somehow misplaced the original high-school leaves I took to Germany with me the notes have nevertheless grown with the passing years. These years have brought me the good things years can bring: marriage, family and career; a good life gratefully lived, details of which interest me and mine but are mostly unremarkable as seen from the outside. A life however can take strange turns, reprising earlier themes. I had become an industrial building construction engineer for a living (and,

[3]The citation is now unfortunately long lost.

appropriately enough, had most lately added to the notes a mathematical justification of the standard industrial building construction technique to measure the resistance to ground of a new building's electrical grounding system[4]), when at a juncture between construction projects an unexpected opportunity arose to pursue graduate work in engineering at Virginia Tech, courtesy (indirectly, as it developed) of a research program not of the United States Army as last time but this time of the United States Navy. The Navy's research problem turned out to be in the highly mathematical fields of theoretical and computational electromagnetics. Such work naturally brought a blizzard of new formulas, whose proofs I sought or worked out and, either way, added to the notes—whence the manuscript and, in due time, this book.

The book follows in the honorable tradition of Courant's and Hilbert's 1924 classic *Methods of Mathematical Physics* [38]—a tradition subsequently developed by, among others, Boas, [25], Jeffreys and Jeffreys [88], Arfken and Weber [6], and Weisstein[5] [178]. The present book's chief intended contribution to the tradition lies in its applied-level derivations of the many results it presents. Its author always wanted to know why the Pythagorean theorem was so. The book is presented in this spirit.

A book can follow convention or depart from it; yet, though occasional departure might render a book original, frequent departure seldom renders a book good. Whether this particular book is original or good, neither or both, is for the reader to tell, but in any case the book does both follow and depart. Convention is a peculiar thing: at its best, it evolves or accumulates only gradually, patiently storing up the long, hidden wisdom of generations past; yet herein arises the ancient dilemma. Convention, in all its richness, in all its profundity, can, sometimes, stagnate at a local maximum, a hillock whence higher ground is achievable not by gradual ascent but only by descent

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[4]The resistance-to-ground technique is too specialized to find place in this book.

[5]Weisstein lists results encyclopedically, alphabetically by name. I organize results more traditionally by topic, leaving alphabetization to the book's index, that readers who wish to do so can coherently read the book from front to back.

There is an ironic personal story in this. As children in the 1970s, my brother and I had a 1959 World Book encyclopedia in our bedroom, about twenty volumes. The encyclopedia was then a bit outdated (in fact the world had changed tremendously during the fifteen or twenty years following 1959, so the book was more than a bit outdated) but the two of us still used it sometimes. Only years later did I learn that my father, who in 1959 was fourteen years old, had bought the encyclopedia with money he had earned delivering newspapers daily before dawn, *and then had read the entire encyclopedia, front to back.* My father played linebacker on the football team and worked a job after school, too, so where he found the time or the inclination to read an entire encyclopedia, I'll never know. Nonetheless, it does prove that even an encyclopedia can be read from front to back.

first—or by a leap. Descent risks a bog. A leap risks a fall. One ought not run such risks without cause, even in such an inherently unconservative discipline as mathematics.

Well, the book does risk. It risks one leap at least: it employs hexadecimal numerals.

This book is bound to lose at least a few readers for its unorthodox use of hexadecimal notation ("The first primes are $2, 3, 5, 7, \mathrm{0xB}, \ldots$"). Perhaps it will gain a few readers for the same reason; time will tell. I started keeping my own theoretical math notes in hex a long time ago; at first to prove to myself that I could do hexadecimal arithmetic routinely and accurately with a pencil, later from aesthetic conviction that it was the right thing to do. Like other applied mathematicians, I've several own private notations, and in general these are not permitted to burden the published text. The hex notation is not my own, though. It existed before I arrived on the scene and, since I know of no math book better positioned to risk its use, I have with hesitation and no little trepidation resolved to let this book use it. Some readers will approve; some will tolerate; undoubtedly some will do neither. The views of the last group must be respected, but in the meantime the book has a mission; and crass popularity can be only one consideration, to be balanced against other factors. The book might gain even more readers, after all, had it no formulas, and painted landscapes in place of geometric diagrams! I like landscapes, too, but anyway you can see where that line of logic leads.

More substantively: despite the book's title and despite the brief philosophical discussion in its chapter 1, adverse criticism from some quarters for lack of rigor is probably inevitable; nor is such criticism necessarily improper from my point of view. Still, serious books by professional mathematicians tend to be *for* professional mathematicians, which is understandable but does not always help the scientist or engineer who wants to use the math to model something. The ideal author of such a book as this would probably hold two doctorates: one in mathematics and the other in engineering or the like. The ideal author lacking, I have written the book.

So here you have my old high-school notes, extended over forty years—years that include professional engineering practice and university study, research and teaching—now partly typed and revised for the first time as a LaTeX manuscript. Where this manuscript will go in the future is hard to guess. Perhaps the revision you are reading is the last. Who can say? The manuscript met an uncommonly enthusiastic reception at Debconf 6 [46] May 2006 at Oaxtepec, Mexico—a reception that, as far as it goes, augurs well for the book's future at least. In the meantime, if anyone should challenge you to prove the Pythagorean theorem on the spot, why, whip this book out and turn to chapter 1. That should confound 'em.

THB

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Index

$'$, 78, 511
0 (zero), 21
    division by, 97
    matrix, 340
    vector, 340, 376
0^- and 0^+ as limits of integration, 664
1 (one), 21, 71, 341
    Fourier transform of, 625
2π, 52, 71
    as a symbol, 742
    calculating, 271
    continued-fraction representation of, 178
I channel, 639
LU decomposition, 378
Q channel, 639
QR decomposition, 451
    inverting a matrix by, 458
T as the cyclic period, 579
Δ, 137
Γ, 714
Λ as the triangular pulse, 583
Ω as the Gaussian pulse, 587, 644, 685
Π as the rectangular pulse, 583
Ψ as the raised-cosine pulse, 583
$\approx$, 63, 103
δ, 97, 231, 337, 514
ϵ, 97, 514
$\equiv$, 25
$\exists$, 595
$\forall$, 595
$\in \mathbb{R}$, 595
$\in \mathbb{Z}$, 29
$\in$, 29, 52, 160, 595
λ as the wavelength, 582
$\langle\cdot\rangle$, 681
$\leftarrow$, 25
$\ll$ and $\gg$, 97
$\mathbb{C}$, 160
$\mathbb{Q}$, 160
$\mathbb{R}$, 160
$\mathbb{Z}$, 160
0x, 743
μ, 681
∇ (del), 535
$\neg$, 595
$\nexists$, 595
ν as the cyclic frequency, 579
ω as the angular frequency, 579
∂ and d, 107
π, 741, 742
ρ, 555
σ, 681
$\subset$, 160
$\vee$, 595
$\wedge$, 595
d and ∂, 107
d, warped or partial, 108
$d\ell$, 229
dz, 253, 254
e, 129
i, 64
k as the spatial angular frequency, 582
n-dimensional vector, 330, 495
nth root
    calculation of by Newton-Raphson, 126
nth-order expression, 313
u, 230
!, 28
!!, 707
a priori probability, 678

reductio ad absurdum, 154, 159, 403
$\mathscr{F}$, 617
$\mathscr{L}$, 663
16th century, 313
19th century, 307
4th century B.C., 11

AA (angle-angle), 59
AAA (angle-angle-angle), 59
AAS (angle-angle-side), 58
Abel, Niels Henrik (1802–1829), 733
abruptness, 585, 621
absolute integral, 619, 621, 659
absolute rate, 116
absolute value, 65
abstraction, 377
accountant, 514
accretion, 211
accuracy, 709
active region, 340, 343
addition
  of matrices, 333
  of rows or columns, 463
  of vectors, 497
  parallel, 172, 314
  serial, 172
  series, 172
addition operator
  downward multitarget, 360
  elementary, 347
  leftward multitarget, 361
  multitarget, 359
  rightward multitarget, 361
  upward multitarget, 360
addition quasielementary, 359
  row, 360
addressing a vector space, 398
adduction, 16
adjoint, 336, 463, 487
  of a matrix inverse, 353
aeronautical engineer, 430
aileron, 430
air, 533, 540, 704
aircraft, 705
Alexandria, Heron of (1st cent. A.D.), 61
algebra, 21
  fundamental theorem of, 170
  higher-order, 313
  linear, 329
  of the vector, 495
algorithm, 163
  conjugate-gradient, 727
  Gauss-Jordan, 384
  implementation of from an equation, 451
all-or-nothing attempt, 696
alternate form, 297
alternating signs, 262, 605
altitude, 55, 59
AMD, 386
amortization, 286
amplitude, 75, 495
amplitudinity, 75
analytic continuation, 245
analytic function, 69, 245
  localized, 647
analyticity, 585
  of the gamma function, 722
Anaxagoras (500–428 B.C.), 109
Anaximander (fl. 570 B.C.), 109
Anaximenes (fl. 550 B.C.), 109
angle, 52, 54, 55, 71, 82, 501, 504, 554
  congruent, 21
  double, 82
  half, 82
  hour, 82
  interior, 52
  of a polygon, 53
  of a triangle, 52
  of rotation, 80
  right, 71
  square, 71
  sum of, 52
angle-angle (AA), 59
angle-angle-angle (AAA), 59
angle-angle-side (AAS), 58
angle-side-angle (ASA), 58
angular frequency, 579, 639
antelope, 534
antenna

parabolic, 524
satellite dish, 524
antiderivative, 211, 281
and the natural logarithm, 282
guess at, 286
of a product of exponentials, powers and logarithms, 310
antiquity, 109, 313
applied mathematics, 2, 231, 240
concepts of, 161
foundations of, 329
approximation
to a square wave, 575
approximation to an arbitrary function, 661
approximation to an arbitrary pulse, 621
approximation to first order, 262
Aquinas, St. Thomas (1225–1274), 13, 111, 732
arbitrary function, 661
arbitrary pulse, 621
arc, 71
arccosine, 71, 143
derivative of, 150
in complex exponential form, 144
Archimedes (287–212 B.C.), 109
arcsine, 71, 143
derivative of, 150
in complex exponential form, 144
arctangent, 71, 143
derivative of, 150
in complex exponential form, 144
area, 1, 21, 51, 220, 555, 594
enclosed by a contour, 542
of a circle, 223
of a sphere's surface, 225
of a triangle, 51, 59
surface, 223
unit, 583
within a parabola, 220
arg, 65
Argand domain and range planes, 246
Argand plane, 65, 137
Argand, Jean-Robert (1768–1822), 65
argument
real, 634
Aristotle (384–322 B.C.), 13, 109, 151
arithmetic, 10, 21, 109, 729
exact, 386, 401, 413, 468, 476
exact and inexact, 432
formal, 3
fundamental theorem of, 156
modular, 191
of matrices, 333
arithmetic mean, 176
arithmetic series, 30
arm, radial, 138
Arnold, D. N., 752
articles "a" and "the", 431
artifice, 190
artillerist, 429
artillery, 307
ASA (angle-side-angle), 58
assertion, 55, 164
assignment, 25
associativity, 21
additive, 21
multiplicative, 21
nonassociativity of the cross product, 504
of convolution, 635, 651
of matrix multiplication, 333
of unary linear operators, 538
assumption, 154, 159
asymmetry, 446
asymptotic series, 705, 708
attempt
all-or-nothing, 696
failed, 658
successful, 661
attitude, 56
autocorrelation, 635
autoderivative, 644
automobile, 505
autotransform, 643, 644
autotransform pair, 642
average, 175
axes, 78
changing, 78

invariance under the reorientation of, 501, 502, 535, 550, 551
reorientation of, 498
rotation of, 78, 495
axiom, 3
axiomatic method, 3
axis, 89
of a cylinder or circle, 507
visual, 247
axle, 509
azimuth, 429, 507, 508, 703

Babylonian science, 161
balanced expression of convolution, 631, 651
band confinement, 620
barometric pressure, 540
baroquity, 325
baseball, 580
Basel problem, 601
BASIC, 761
basis, 478, 510
complete, 478
constant, 506, 559
converting to and from, 478
cylindrical, 507
orthogonal, 505
secondary cylindrical, 508
spherical, 507
variable, 506
battle, 429
battlefield, 429
bearing, 700
bell curve, 644, 687
Bell, John L. (1945–), 7
belonging, 29, 52
Bertrand Russell (1872–1970), 8, 11, 111
bias, 178
binomial theorem, 101
bisection, 524
bit, 178, 339, 386, 433
repetition of, 180
bit sequence
canonical, 200
bit-sequence representation, 180
bivouac, 762
Black, Thaddeus H. (1967–), 761
blackbody radiation, 46
block, wooden, 98
blockade, 406
bond, 116
borrower, 286
botanist, 737
bound, 202
spiral inward toward, 194
bound on a power series, 262
boundary condition, 286
boundary element, 545, 547
boundedness, 191
bounding, 202
Bourbaki, Nicolas (1935–), 161
box, 21
Box, G. E. P. (1919–2013), 704
Box-Muller transformation, 704
bracket, 709
branch, 737
branch point, 63, 247
strategy to avoid, 249
brick, 172, 652
bridge, 4, 111, 731
Brouwer, Luitzen Egbertus Jan (1881–1966), 110
Bryan, George Hartley (1864–1928), 498
building construction, 434
Bulmer, M. G. (1931–), 696
Byss and Abyss, the, 729

C and C++, 23, 25
C++, 731
calculus, 95, 205
fundamental theorem of, 211, 546
of the vector, 533
the two complementary questions of, 95, 205, 211
vector, definitions and identities of, 549
cannon, 429
canonical form
of the bit-sequence representation, 200

of the continued-fraction representation, 199
Cantor, Georg (1845–1918), 11, 111, 597, 732
capacitor, 151
capital
Greek, 587
card, 678
Cardan rotations, 498
Cardan, Girolamo (also known as Cardano or Cardanus, 1501–1576), 313, 498
carriage wheel, 429
case
at the corner, 320
at the edge, 15, 319, 376
special, 164
Cauchy's impressed residue theorem, 752
Cauchy's integral formula, 253, 288
Cauchy, Augustin Louis (1789–1857), 253, 288, 733
Cauchy-Schwarz inequality, 446
caveman, 534
CDF (cumulative distribution function), 679
chain rule, derivative, 118
change of variable, 25
change, rate of, 95
channel, 639
characteristic polynomial, 470, 485, 487
checking an integration, 228
checking division, 228
choice of wooden blocks, 98
Cholesky, André-Louis (1875–1918), 494
Chrysippus (280–206 B.C.), 109
Churchill, Sir Winston (1874–1965), 729
Cicero (106–43 B.C.), 762
circle, 71, 82, 137, 507
area of, 223
secondary, 508
travel about, 138
unit, 71
circuit, 430
circuit theory, 731
circular paraboloidal coordinates, 529
circular reasoning, 165
circulation, 542, 547
cis, 142
city street, 430
cleverness, 61, 234, 321, 560, 644, 752
climate, 678
clock, 82
closed analytic form, 310
closed contour
about a multiple pole, 259
closed contour integration, 229
closed form, 310
closed surface integration, 227
clutter, notational, 511, 745
coïncident properties of the square matrix, 468
coefficient, 329
Fourier, 589, 654
inscrutable, 316
matching of, 42
metric, 554
nontrivial, 376, 472
unknown, 286
coin, 696
collated pulse, 654
column, 330, 387, 425, 453
addition of, 463
null, 462
null, appending a, 421
orthonormal, 457
scaled and repeated, 462
spare, 391, 412
column operator, 335
column rank, 438
full, 410, 412
column vector, 403, 444
combination, 98
properties of, 99, 100
combinatorics, 98
properties of, 99, 100
common divisor, greatest, 162, 163
common multiple, least, 169
commutation
hierarchy of, 349
of elementary operators, 349

of matrices, 350
of the identity matrix, 345
commutivity, 21
additive, 21
multiplicative, 21
noncommutivity of matrix multiplication, 333
noncommutivity of the cross product, 503
noncommutivity of unary linear operators, 537
of convolution, 635, 651
of the dot product, 501
summational and integrodifferential, 215
compactness, 585
complementary variables of transformation, the, 617
complete basis, 478
completing the square, 26, 644
complex conjugation, 67
complex contour
about a multiple pole, 259
complex coordinate, 497
complex exponent, 140
complex exponential, 129, 589
and de Moivre's theorem, 141
derivative of, 146
infinitesimally graded, 617
inverse, derivative of, 146
Laplace transform of, 665
properties of, 145
sampled, 596
superposition of, 575
complex number, 64, 92, 160
actuality of, 150
being a scalar not a vector, 330
conjugating, 67
imaginary part of, 66
magnitude of, 65
multiplication and division, 66, 92
phase of, 65
real part of, 66
complex number in nature, 150
complex plane, 65
complex power, 104
complex trigonometrics, 141
inverse, 143
complex variable, 16, 114, 751
component
of a vector field, 533
components of a vector by subscript, 511
composite number, 153
compositional uniqueness of, 154
compromise, 585
computer, 705
pseudorandom-number generator of, 699
computer memory, 432, 451
computer processor, 386
computer register, 432
concepts of applied mathematics, 161
concepts of mathematics, 161
concert hall, 48
concision, 510
concluding remarks, 729
condition, 469, 476
of a scalar, 477
conditional convergence, 216
cone
volume of, 224
congruence, 55
conjecture, 439
conjugate, 64, 67, 150, 336
Fourier transform of, 623
of a Fourier series, 599
conjugate transpose, 336
of a matrix inverse, 353
conjugate-gradient algorithm, 727
conjugation, 67
quadratic, 27
connection, deep, 602
connotation, 75
consciousness, 11
consensus
vacillating, 112
constant, 47
Fourier transform of, 625
constant expression, 25
constant, indeterminate, 47

constraint, 322
    that a quantity be an integer, 191
construction of an arbitrary function, 661
construction of an arbitrary pulse, 621
constructive function, 656
continued fraction, 178
    extraction of the terms of, 180
    extraction of the terms of from a quadratic root, 184
    extraction of the terms of from a rational number, 198
    extraction of the terms of from an irrational, 181
    extraction of the terms of from an rational, 180
    quadratic repetition in, 188
    right-to-left computation of, 179
    truncation of, 178
continued-fraction
    canonical form of, 199
    construction of the ratio from, 201
    estimation and bounding by, 202
    relative primes in, 201
continuity, 109
continuous and discrete systems, 731
continuous waveform, 597
continuum, 12
contour, 230, 247, 507
    closed, 547
    closed planar, 542
    complex, 254, 258, 289, 608, 644
    complex, about a multiple pole, 259
    derivative product rule along, 552
contour infinitesimal, 555, 571
contour integration, 229, 571
    closed, 229
    closed complex, 253, 288, 608, 646
    complex, 254, 644
    of a vector quantity, 230, 571
contractor, 434
contradiction, proof by, 154, 159, 403
control, 430
control surface, aeronautical, 430
convention, 669, 741
convergence, 92, 215, 239, 733
    conditional, 216
    domain of, 267
    improvement of, 727
    lazy, 709
    related to the raised cosine-rolloff pulse, 619
    slow, 267
convolution, 631, 651, 652, 680
    associativity of, 635, 651
    balanced expression of, 631, 651
    by Heaviside's unit step, 648
    commutivity of, 635, 651
    definition of, 652
    Fourier transform of, 631
    in cyclic frequencies, 671
    unbalanced expression of, 631, 651
coordinate, 495
    complex, 497
    primed and unprimed, 511
    real, 497
coordinate grid
    parabolic, 526
coordinate rotation, 89
coordinates, 89, 510, 554
    circular paraboloidal, 529
    cyclic progression of, 503
    cylindrical, 89, 554
    isotropic, 521
    logarithmic cylindrical, 521
    parabolic, 522, 570
    parabolic cylindrical, 529
    parabolic, in two dimensions, 525
    parabolic, isotropy of, 528
    parabolic, properties of, 528
    rectangular, 71, 89
    relations among, 90
    special, 522
    spherical, 89, 554
cord, surveyor's, 61
corner, 585
corner case, 320
corner value, 314
correlation, 631, 694
    Fourier transform of, 631
    inference of, 694

correlation coefficient, 694
cosine, 71, 501
  derivative of, 146, 148
  Fourier transform of, 643
  in complex exponential form, 141
  Laplace transform of, 665
  law of, 87
cost function, 437
countability, 580
counterexample, 333
Courant, Richard (1888–1972), 5
Courant-Hilbert-Shilov perspective, the, 5
course, 731
Cramer's rule, 468
Cramer, Gabriel (1704–1752), 468
crankshaft, 579
creativity, 307
cross product, 501
  nonassociativity of, 504
  noncommutivity of, 503
  perpendicularity of, 504
cross-derivative, 279
cross-directional curl, 544
cross-directional derivative, 544
cross-section
  parabolic, 524
cross-term, 279
crosswind, 505
cryptography, 153
cubic expression, 25, 172, 313, 314
  roots of, 317
cubic formula, 317
cubing, 318
cue, verbal, 580
cumulative distribution function, 679
  estimation of, 707
  numerical calculation of, 688
  of the normal distribution, 688, 705
cumulative normal distribution function, 688
curl, 542
  cross-directional, 544
  directional, 542, 547
  in cylindrical coordinates, 560
  in nonrectangular coordinates, 568
  in spherical coordinates, 563
  of a curl, 550
  of a gradient, 551
current, electric, 75, 151
customer, 700
cycle, 580
  integration over a complete, 589
cyclic frequency, 579, 671
  convolution in, 671
  Fourier transform in, 671
  inverse Fourier transform in, 671
cyclic progression of coordinates, 503
cylinder, 507
  parabolic, 529
cylinder function, 727
cylindrical basis, 507
cylindrical coordinates, 89, 495, 554
  integration in, 646
  parabolic, 529

datum, 433, 434
day, 82
days of the week, 535
de Moivre's theorem, 92, 141
  and the complex exponential, 141
Debian, 17
Debian Free Software Guidelines, 17, 751
deck of cards, 678
decomposition
  LU, 378
  QR, 451
  diagonal, 473
  differences between the Gram-Schmidt and Gauss-Jordan, 454
  eigenvalue, 473
  Gauss-Jordan, 378
  Gram-Schmidt, 451
  Gram-Schmidt, inverting a matrix by, 458
  orthonormalizing, 451
  Schur, 479
  singular-value, 491
deduction, 16
deep connection, 602

definite integral, 228
  to represent the gamma function, 714
definition, 3, 104, 161, 231
  abstract, 3
  of vector calculus, 549
definition notation, 25
degenerate linear system, 410
degenerate matrix, 409, 421
degree of freedom, 429
del (∇), 535
delay, 626, 652
delta function, Dirac, 231, 652
  alternative to, 583
  as implemented by the Gaussian pulse, 647
  Fourier transform of, 625
  implementation of, 583
  implementation of, subtle, 587
  response to, 631
  sifting property of, 231, 625, 648, 652
delta, Kronecker, 337, 514
  properties of, 516
  sifting property of, 337
Democritus (born c. 458 B.C.), 109
demotion, 407
denominator, 38, 293
  vanishing, in the Fourier transform of a raised cosine-rolloff pulse, 621
density, 219
  spectral, 636
density function, probability, 679
dependence, 694
dependent element, 425
dependent variable, 106
Derbyshire, John (1945–), 113
derivation, 1
derivative, 95
  balanced form of, 105, 114
  chain rule for, 118
  constant, 540
  cross-, 279
  cross-directional, 544
  definition of, 105
  directional, 538
  Fourier transform of, 629
  higher, 120
  higher-order, 114
  Jacobian, 372, 436, 443
  Laplace transform of, 664
  Leibnitz notation for, 106
  logarithmic, 116, 132
  logarithmic of the natural exponential, 132
  manipulation of, 117
  Newton notation for, 105
  nonexistent, 114
  of z^a/a, 282
  of a complex exponential, 146
  of a field, 533
  of a field in cylindrical coordinates, 559
  of a field in cylindrical coordinates, second-order, 561
  of a field in spherical coordinates, 563
  of a field in spherical coordinates, second-order, 566
  of a field, nonrectangular, 557
  of a field, second-order, 550
  of a Fourier transform, 629
  of a function of a complex variable, 114
  of a rational function, 300
  of a trigonometric, 148
  of a unit basis vector, 557
  of an inverse trigonometric, 150
  of arcsine, arccosine and arctangent, 150
  of sine and cosine, 146
  of sine, cosine and tangent, 148
  of the natural exponential, 130, 148
  of the natural logarithm, 133, 150
  of the sine-argument function, 604
  of z^a, 115
  partial, 108, 535
  product rule for, 118, 120, 284, 373, 548, 552
  second, 114, 121

trial, 307
unbalanced form of, 105, 114
with respect to position, 534
derivative pattern, 120
derivative product
pattern of, 120
determinant, 459
and the elementary operator, 464
definition of, 460
inversion by, 467
of a matrix inverse, 466
of a product, 466
of a unitary matrix, 466
product of two, 466
properties of, 461
rank-n, 460
zero, 465
determination of primality, 158
deviation, 440
device, 732
DFSG (Debian Free Software Guidelines), 17, 751
diag notation, the, 359
diagonal, 1, 54, 55, 71
main, 338, 485
three-dimensional, 55
diagonal decomposition, 473
diagonal matrix, 358, 359
diagonalizability, 474
diagonalizable matrix, 488
diagonalization, 473
die, 678
differentiability, 115
differential equation, 108, 286
ordinary, 108
partial, 108
solution of by the Laplace transform, 666
solution of by unknown coefficients, 286
differentiation
analytical versus numeric, 229
Fourier transform of, 629
digamma function, 723
dimension, 330, 578, 579, 639, 669
visual, 247
dimension-limited matrix, 340
dimensionality, 78, 338, 414
dimensionlessness, 71, 582
Dirac delta function, 231, 652
alternative to, 583
as implemented by the Gaussian pulse, 647
Fourier transform of, 625
implementation of, 583
implementation of, subtle, 587
response to, 631
sifting property of, 231, 625, 648, 652
Dirac delta pulse train, 594
Fourier coefficients of, 594
Fourier transform of, 643
Dirac, Paul (1902–1984), 231
direction, 75, 495
directional curl, 542, 547
directional derivative, 538
in cylindrical coordinates, 559
in spherical coordinates, 563
directrix, 523
Dirichlet, J. Peter Gustav Lejeune (1805–1859), 591, 733
disconsolate infinity, 110
discontinuity, 12, 230, 585
in the square pulse, 585
discontinuous waveform, 597, 610
discovery, 732
discrete and continuous systems, 731
discreteness, 109
discretization, 595
dish antenna, 524
displacement, 75, 555
displacement infinitesimal, 555, 571
display, 627
distance, 77
distinct pole, 297
distribution, 679
conversion between two, 684, 704
default, 689, 699
exponential, 700
Gaussian, 687

Maxwell, 704
normal, 685, 687
normal, proof of, 686
Poisson, 700
Rayleigh, 702
uniform, 699
distribution function, cumulative, 679
distributivity, 21
of unary linear operators, 537
divergence, 540, 545, 714
in cylindrical coordinates, 560
in nonrectangular coordinates, 566
in spherical coordinates, 563
of a curl, 551
of a gradient, 550
of the sum of plain inverses, 602
related to the square pulse, 619
divergence theorem, 545
divergence to infinity, 63
divergenceless field, 551
divergent series, 270
dividend, 38
division, 66, 92
by matching coefficients, 42
checking, 228
trial, 158, 165
division by zero, 97
divisor, 38
greatest common, 162, 163
dollar, 581
domain, 63, 267, 425
sidestepping a, 269
time and frequency, 617, 666
transform, 617, 666
domain contour, 247
domain neighborhood, 245
dominant eigenvalue, 475, 476
dot product, 444, 501
abbrevated notation for, 511
commutivity of, 501
double angle, 82
double integral, 219
ill-behaved, 218
double pole, 294, 300
double root, 320
doubt, 167
down, 71, 430
downstairs, 534
downward multitarget addition operator, 360
driving vector, 423, 425, 431
duality, 624, 626
dull pulse, 583
dummy variable, 28, 213, 288, 617
duty cycle, 593
vanishing, 594

east, 71, 429, 507
edge
inner and outer, 547
edge case, 15, 319, 376
edge value, 231
efficiency, 705
efficient implementation, 451
effort, 57
eigensolution, 472
count of, 473
impossibility to share, given independent eigenvectors, 473
of a matrix inverse, 472
repeated, 475, 486
eigenvalue, 459, 470
distinct, 472, 474
dominant, 475, 476
large, 476
magnitude of, 476
of a matrix inverse, 471
perturbed, 486
real, 488
repeated, 473, 486
shifting of, 472
small, 476
zero, 471, 476
eigenvalue decomposition, 473
eigenvalue matrix, 473
eigenvector, 470, 471
generalized, 487
independent, 474
linearly independent, 472
of a matrix inverse, 472

orthogonal, 488
repeated, 475, 486
Einstein notation, 513, 556
Einstein's summation convention, 513, 556
Einstein, Albert (1879–1955), 513
electric capacitor, 151
electric circuit, 430
electric current, 75, 151
electric tension or potential, 75, 151
electrical engineer, 430, 612, 631, 636
electromagnetic power, 504
electromagnetics, 731
electronic signal, 585
electronic signaling, 636
electronics, 731
elegance, 705
element, 330
free and dependent, 425
interior and boundary, 545, 547
elementary algebra, 21
elementary function, 711
elementary geometry, 21
elementary operator, 346
addition, 347, 352
and the determinant, 464
combination of, 350
commutation of, 349
expansion of, 350
interchange, 346, 351
inverse of, 346, 350
invertibility of, 348
scaling, 347, 352
sorting of, 349
two, of different kinds, 349
elementary similarity transformation, 378
elementary vector, 345, 403, 444
left-over, 405
elevation, 429, 507
elf, 64
embedded control, 705
embedded device, 705
empirical probability, 678
empty set, 376
end, justifying the means, 402
energy, 75, 639
kinetic, 108
potential, 108
energy spectral density, 636
engine, 579
engineer, 161, 430, 612, 631, 669, 675
ensemble, 691
entire function, 249, 267, 275, 646, 723
Epicurus (341–271 B.C.), 109
epistemology, 8, 13
epsilon, Levi-Civita, 514
properties of, 516
equation
simultaneous linear system of, 333, 422
solving a set of simultaneously, 80, 333, 422
superfluous, 423
equator, 225
equidistance, 523
error
due to rounding, 432
forestalled by rigor, 13
in the solution to a linear system, 477
error bound, 262, 707
error, rounding, 705
essential singularity, 63, 250, 277
estimation, 202
estimation of statistics, 690
etymology, 165
Euclid (325–265 B.C.), 54, 55, 109, 154, 163
Euclid's algorithm, 163
Euclid's assertion, 164
Euclid's conclusion, 164
Eudoxus (408–355 B.C.), 109
Euler rotations, 500
Euler's formula, 137
curious consequences of, 140
Euler, Leonhard (1707–1783), 137, 140, 216, 500, 601, 727
Euler-Mascheroni constant, 727
European Renaissance, 14
evaluating when, 98
evaluation, 120

even function, 272
  Fourier transform of, 639
even inverse square
  sum of, 602
even square
  inverse, sum of, 602
evenness, 178, 354
exact arithmetic, 386, 401, 413, 432, 468, 476
exact matrix, 401
exact number, 401
exact quantity, 401
exactly determined linear system, 422
exercises, 175, 233, 305
existence, 302
expansion of $1/(1-z)^{n+1}$, 236
expansion point
  shifting of, 241
expectation, 678
expected value, 681
experience, 307
experimental measurement, 662
exponent, 32, 49
  complex, 140
  floating-point, 433
  sum of, 35
exponent, floating-point, 386
exponential, 49, 140
  approximation of to first order, 262
  general, 133
  integrating a product of a power and, 308
  Laplace transform of, 665
  natural, error of, 267
  resemblance of to x^∞, 136
exponential decay
  Fourier transform of, 642
exponential distribution, 700
exponential, complex, 129, 589
  and de Moivre's theorem, 141
  sampled, 596
exponential, natural, 129
  compared to x^a, 134
  derivative of, 130, 148
  existence of, 129
  Fourier transform of, 642
  logarithmic derivative of, 132
exponential, real, 129
expression
  Pythagorean, 303
extended operator, 339, 341, 458
  decomposing, 396
extension, 14
extra column, 391, 412
extremum, 120, 251
  global, 605
  local, 606
  of the sine integral, 606
  of the sine-argument function, 605
eye of the mind, 110

factor
  of integration, 109, 207
  prime, 167
  truncation of, 394
factorial, 28
  !!-style, 707
  as related to the gamma function, 715
factorization, 25
  QR, 451
  full-rank, 412, 439, 455
  Gauss-Jordan, 378
  Gram-Schmidt, 451
  orthonormalizing, 451
  prime, 154
failed attempt, 658
failure, 658
failure of a mechanical part, 700
false try, 237, 608
family of solutions, 431
fast function, 134
faux rigor, 99
feedback, 307
Ferrari, Lodovico (1522–1565), 313, 321
Feynman, Richard P. (1918-1988), 696
field, 533
  curl of, 542
  derivative of, 533

derivative of, in cylindrical coordinates, 559
derivative of, in spherical coordinates, 563
derivative of, nonrectangular, 557
derivative product rule for, 548, 552
directional derivative of, 538
divergence of, 540
divergenceless, 551
gradient of, 538
irrotational, 551
second-order derivative of, 550
solenoidal, 551
source-free, 551
filling of a pattern, 664
final value
of the sine integral, 606
final-value theorem, 668
financial unit, 76
finitude, 658
first-order approximation, 262
first-order Taylor expansion, 104
Fisher, R. A. (1890–1962), 696
flattening, 684
flaw, logical, 156
floating-point exponent, 433
floating-point infinity, 433
floating-point number, 386, 476
floating-point register, 432, 601
floating-point zero, 433
floor, 162, 534
flux, 540, 545
focus, 523
football, 540
forbidden point, 243, 246
force, 75
form, alternate, 297
formal arithmetic, 3
formal mathematical rigor, 3
formal parameter, 108
formalism, 7, 214
Fortran, 29
foundational program, 161
foundational schools of mathematical thought, 11
foundations of mathematics, 7, 161, 731
fourfold integral, 219
Fourier autotransform, 643, 644
Fourier autotransform pair, 642
Fourier coefficient, 589, 654
as connected to a function's mean square, 599
derivation of formula for, 589
of the Dirac delta pulse train, 594
of the rectangular pulse train, 592
of the square wave, 592
real and imaginary parts of, 598
recovery of, 589
Fourier series, 575, 589
as multiplied by its conjugate, 599
in trigonometric form, 598
linearity of, 595
sufficiency of, 595
Fourier transform, 615
applications of, 651
comparison of against the Laplace transform, 666
differentiation of, 629
dual of, 624, 626
example of, 619
frequency-shifted, 626
in cyclic frequencies, 671
in primitive guise, 575
independent variable and, 617
inverse, 617
linearity of, 629
metadual of, 626
of a complex conjugate, 623
of a constant, 625
of a convolution, 631
of a correlation, 631
of a delayed function, 626
of a derivative, 629
of a Dirac delta pulse train, 643
of a function whose argument is complex, 634
of a level and ramp, 661
of a product, 631
of a raised cosine-rolloff pulse, 619
of a raised-cosine pulse, 619

of a ramp and level, 658, 661
of a shifted function, 629
of a sinusoid, 643
of a square pulse, 619
of a triangular pulse, 619
of an exponential decay, 642
of an irregular step, 660
of an irregular triangular pulse, 656
of an odd or even function, 639
of an right-triangular pulse, 659
of differentiation, 629
of integration, 648
of selected functions, 640
of the Dirac Delta, 625
of the Heaviside unit step, 642
of the natural exponential, 642
of the sine-argument function, 640
properties of, 623, 664, 665
real and imaginary parts of, 623
reversing the independent variable of, 624
scaling of, 629
shifting of, 629
spatial, 219, 669
superpositional property of, 629
symmetry of, 623, 675
Fourier transform pair, 617, 641, 664, 666, 674
Fourier's equation, 615
Fourier, Jean Baptiste Joseph (1768–1830), 575, 615, 731
fraction, 293
Fraenkel, Abraham (1891–1965), 11
frame offset, 578
free element, 425
freedom, degree of, 429
freeway, 434
Frege, Friedrich Ludwig Gottlob (1848–1925), 8, 729
frequency, 579, 581, 639
angular, 579, 639
cyclic, 579, 671
infinitesimally graded, 617
primary, 576
shifting of, 589
spatial, 582, 639, 669
frequency content, 617, 639
frequency domain, 617, 666
frequency shift, 626
freshman, 731
frontier, 366
Frullani's integral, 307
Frullani, Giuliano (1795–1834), 307
full column rank, 410, 412
full rank, 409
full row rank, 410, 449
full-rank factorization, 412, 439, 455
function, 62, 231
analytic, 69, 245
arbitrary, 661
constructive, 656
elementary or nonelementary, 711
entire, 249, 267, 275, 646
experimentally measurable, 662
extremum of, 120
fast, 134
fitting of, 235
inverse of, 63, 684
linear, 215
localized analytic, 647
meromorphic, 249, 274
nonanalytic, 69
nonlinear, 215
odd or even, 272, 639
of a complex variable, 114
of position, 533
rational, 293
rational, derivatives of, 300
rational, integral of, 298
sample of, 662
single- and multiple-valued, 246, 247
slow, 134
special, 711
sum of, 652
tail of, 661
unreasonable, 246
versatile, 647
fundamental theorem of algebra, 170
fundamental theorem of arithmetic, 156

fundamental theorem of calculus, 211, 546

Gödel, Kurt Friedrich (1906–1978), 13, 729
Göttingen, 733
game of chance, 678
gamma function, 711
  analyticity of, 722
  as related to the factorial, 715
  definite integral representation of, 714
  logarithmic derivative of, 723
  numerical evaluation of, 716
  of a half-integral argument, 715
  pole of, 722
  reciprocal, 723
  reflection of, 719
  residue of, 722
Gauss, Carl Friedrich (1777–1855), 378, 424, 644, 687, 733
Gauss-Jordan algorithm, 384
Gauss-Jordan decomposition, 378
  and full column rank, 412
  differences of against the Gram-Schmidt, 454
  factors K and S of, 412
  inverting the factors of, 393
Gauss-Jordan factorization, 378
Gauss-Jordan factors
  properties of, 395
Gauss-Jordan kernel formula, 424
Gaussian distribution, 687
Gaussian pulse, 587, 644, 687
  properties of, 587
  to implement the Dirac delta by, 647
GCD, 162, 163
gear, 581
general exponential, 133
general identity matrix, 341
general interchange operator, 357
General Public License, GNU, 17
general scaling operator, 358
general solution, 432
general triangular matrix, 362, 485
generality, 424
generalized eigenvector, 487
geometric majorization, 265
geometric mean, 176
geometric series, 45
  majorization by, 265
  variations on, 46
geometrical argument, 4, 566
geometrical intuition, 56
geometrical vector, 75, 495
geometrical visualization, 377, 566
geometry, 10, 21, 51, 109
Gibbs phenomenon, 610
Gibbs, Josiah Willard (1839–1903), 610
giving that, 98
GNU General Public License (GPL), 17
goal post and goal line, 540
golden ratio, the, 182
Goursat, Edouard (1858–1936), 753
GPL, 17
gradient, 538
  in cylindrical coordinates, 559
  in nonrectangular coordinates, 570
  in spherical coordinates, 563
  of a divergence, 550
grading, infinitesimal, 617
Gram, Jørgen Pedersen (1850–1916), 449
Gram-Schmidt decomposition, 451
  differences of against the Gauss-Jordan, 454
  factor Q of, 455
  factor S of, 454
  inverting a matrix by, 458
Gram-Schmidt kernel formula, 456
Gram-Schmidt orthogonal complement, 455
Gram-Schmidt process, 449
grapes, 150
greatest common divisor, 162, 163
Greek alphabet, 747
Greek capital, 587
Greek philosophy, 732
Greenwich, 82
grid, parabolic coordinate, 526
guessing roots, 325
guessing the form of a solution, 286

gunpowder, 429
Gymnasium, 736

Hadamard, Jacques (1865–1953), 114
half
  gamma function of, 715
half angle, 82
Hamilton, William Rowan (1805–1865), 536
Hamming, Richard W. (1915–1998), 13, 205, 240
harmonic mean, 176
harmonic series, 265
headwind, 505
Heaviside unit step function, 230
  convolution by, 648
  Fourier transform of, 642
Heaviside, Oliver (1850–1925), 5, 150, 230, 536
height, 76, 677
Helmholtz, Hermann Ludwig Ferdinand von (1821–1894), 551, 552
Heraclitus (fl. 500 B.C.), 109
Hercules, 110
Hermite, Charles (1822–1901), 336, 487
Hermitian matrix, 487
Heron of Alexandria (1st cent. A.D.), 61
Heron's rule, 59
Hersh, Reuben (1927–2020), 7
hertz, 579, 675
Hessenberg matrix, 487
Hessenberg, Gerhard (1874–1925), 487
hexadecimal, 743
  canonical, 200
hiding, 297
higher-order algebra, 313
higher-order derivative, 114
Hilbert, David (1862–1943), 5, 729
Hildebrand, F. B., 753
homogeneous solution, 431
horizontal run, 71
horse, 429
hour, 82
hour angle, 82
house, 698
Hume, David (1711–1776), 729
hut, 534
hyperbolic arccosine, 143
hyperbolic arcsine, 143, 221
hyperbolic arctangent, 143
hyperbolic cosine, 142
hyperbolic functions, 142
  inverse, in complex exponential form, 144
  properties of, 143
hyperbolic sine, 142
hyperbolic tangent, 142
hyperbolic trigonometrics, 142
hypotenuse, 1, 55
hypothesis, 426
  reverse, 197

identifying property, 12, 21
identity
  additive, 21
  arithmetic, 21
  differential, of the vector, 547
  multiplicative, 21
  of vector calculus, 549
  vector, algebraic, 519
identity matrix, 341, 344
  r-dimensional, 344
  commutation of, 345
  impossibility to promote of, 403
  rank-r, 344
iff, 173, 215, 239
ill-conditioned matrix, 469, 476
imaginary number, 64
imaginary part, 66
  of the Fourier transform, 623
imaginary unit, 64, 137
immovability, 110
implementation
  efficient, 451
imprecise quantity, 401, 476
impressed residue theorem, Cauchy's, 752
improper sequence, 355
improvement of convergence, 727
impulse function, 231
imputed ensemble, 691

incrementation
    infinitesimal, 138
indefinite integral, 228
independence, 376, 410, 694
independent infinitesimal
    variable, 210
independent variable, 106
    Fourier transform and, 617
    lack of, 108
    multiple, 108
    reversal of, 283
    scaling of, 284
indeterminate form, 123
index, 28, 332
    of multiplication, 28
    of summation, 28
    swapping of, 446
index of stray, 697
indictment, 10
induction, 67, 238
    skip-, 190
indulgence, 10, 110
industrial electronics, 731
inequalities
    complex vector, 446
    vector, 446
inequality, 24, 440
    power-related, 36
    Schwarz, 446
inexact arithmetic, 432
inexact quantity, 401
inference of statistics, 690
infinite differentiability, 245
infinite dimensionality, 338
infinite rank, 410
infinite slope, 621
infinitesimal, 96
    and the Leibnitz notation, 106
    displacement, 555, 571
    dropping of when negligible, 253
    independent, variable, 210
    mental image of, 107
    practical size of, 97
    referential, 114
    second- and higher-order, 97
    surface, 233, 555, 571
    vector, 571
    volumetric, 233, 555
infinitesimal factor of integration, 109, 207
infinitesimal grading, 617
infinitesimal incrementation, 138
infinitesimality, 658
infinitude, 658
infinity, 3, 96
    disconsolate, 110
    floating-point, 433
inflection, 121
initial condition, 666
initial-value theorem, 668
inner edge, 547
inner product, 444
inner surface, 545
insight, 578
inspiration, 61
instance, 690
instructor, 731, 736
integer, 28, 31, 160
    composite, 153
    compositional uniqueness of, 154
    prime, 153
    sequence of, 178
    whose value cannot easily be determined in the abstract, 191
integrability, 546
integral, 205
    absolute, 619, 621, 659
    and series summation, 264
    as accretion or area, 206
    as antiderivative, 211
    as shortcut to a sum, 207
    as the continuous limit of a sum, 615
    balanced form, 209
    closed complex contour, 253, 288, 608, 646
    closed contour, 229
    closed surface, 227
    complex contour, 254
    concept of, 205
    contour, 229, 571

definite, 228
double, 219
fourfold, 219
ill-behaved, 218
indefinite, 228
Laplace transform of, 664
magnitude of, 292
multiple, 218
of a rational function, 298
of the sine-argument function, 604
sixfold, 219
surface, 219, 225, 571
swapping one vector form for another in, 546, 547
to represent the gamma function, 714
triple, 219
vector contour, 230, 571
volume, 219
integral equation, 724
integral forms of vector calculus, 545
integral quotient, 162
integral swapping, 218
integrality, 188
integrand, 281
magnitude of, 292
integrated circuit, 111
integration
analytical versus numeric, 229
as summation, 611
by antiderivative, 281
by closed contour, 288, 608, 646
by conversion to cylindrical or polar form, 646
by partial-fraction expansion, 293
by parts, 284, 705
by reversal of the independent variable, 283
by scaling of the independent variable, 284
by substitution, 282
by Taylor series, 310
by the manipulation of a Pythagorean expression, 303
by the manipulation of a Pythagorean nonradical, 306
by the manipulation of a Pythagorean radical, 303
by unknown coefficients, 286
checking of, 228
double, 670
factor of, 109, 207
fourfold, 670
Fourier transform of, 648
infinitesimal factor of, 109, 207
limit of, 207
of a product of exponentials, powers and logarithms, 308
over a complete cycle, 589
sixfold, 670
surface, 670
triple, 669
volume, 669
integration technique, 281, 646
Intel, 386
interchange, 453, 461
refusing an, 412, 451
interchange operator
elementary, 346
general, 357
interchange quasielementary, 357
interest, 116, 286
interior element, 545, 547
internal-combustion engine, 579
interpolation, 663
interval arithmetic, 181
interval, sampling, 621
intuition, 56
intuitive proposition, 12
invariance under the reorientation of axes, 501, 502, 535, 550, 551
inverse, 579
determinant of, 466
existence of, 420
mutual, 420
of a function, 63, 684
of a matrix product, 353
of a matrix transpose or adjoint, 353
rank-r, 418
sum of, 602
uniqueness of, 420

inverse complex exponential
derivative of, 146
inverse Fourier transform, 617
inverse hyperbolic functions
in complex exponential form, 144
inverse square
even, sum of, 602
odd, sum of, 602
sum of, 601, 602
inverse time, 579
inverse trigonometric family of functions, 143
inversion, 350, 417, 418
additive, 21
arithmetic, 21
by determinant, 467
multiplicative, 21, 22
symbolic, 459, 467
invertibility, 63, 438, 596
of the elementary operator, 348
invocation, 165
irrational number, 159
continued-fraction representation of, 181
irreducibility, 3, 159, 162, 202
irregular raised cosine, 663
comparison of against the irregular triangular pulse, 663
irregular step
Fourier transform of, 660
irregular triangular pulse
comparison of against the irregular raised cosine, 663
Fourier transform of, 656
overlapping, 663
irrotational field, 551
isotropy, 521
of parabolic coordinates, 528
iteration, 124, 163, 184, 476
final, 167
which does not alter the result, 182

Jacobi, Carl Gustav Jacob (1804–1851), 108, 372, 735
Jacobian derivative, 372, 436, 443
Japanese yen, 76
jet engine, 509
Jordan, Wilhelm (1842–1899), 378, 424
jump, 585
junior, 731
justification, 161

Kant, Immanuel (1724–1804), 151, 729
Kelvin, Lord (1824–1907), 575
kernel, 423, 669
alternate formula for, 426
Gauss-Jordan formula for, 424
Gram-Schmidt formula for, 456
kernel matrix, 423
converting between two of, 428
kernel space, 424, 428
kilogram, 77
kinetic energy, 108
Klein, C. Felix (1849–1925), 733
knee, 7
knight, 762
Korté, Herbert, 7
Kronecker delta, 337, 514
properties of, 516
sifting property of, 337
Kronecker, Leopold (1823–1891), 337, 514

l'Hôpital's rule, 122, 274, 621
l'Hôpital, Guillaume de (1661–1704), 122
labor union, 434
lack of time, 13
Lagrange, Joseph-Louis (1736–1813), 196
language, natural, 729
Laplace transform, 651, 663
comparison of against the Fourier transform, 666
higher-order properties of, 664
initial and final values by, 668
of a convolution, 664
of a derivative, 664
of an integral, 664
ramping property of, 664
solving a differential equation by, 666
Laplace transform of a sinusoid, 665
Laplace transform of an exponential, 665

Laplace, Pierre-Simon (1749–1827), 550, 651, 663
Laplacian, 550
large-argument form, 705
Latin, 38, 652, 762
Laurent series, 37, 275
Laurent, Pierre Alphonse (1813–1854), 37, 275
law of cosines, 87
law of sines, 86
lazy convergence, 709
LCM, 169
least common multiple, 169
least squares, 434
least-squares solution, 436
lecture, 696
leftward multitarget addition operator, 361
leg, 1, 55
Legendre polynomial, 727
Legendre, Adrien-Marie (1752–1833), 727
Leibnitz notation, 106, 107, 210
Leibnitz, Gottfried Wilhelm (1646–1716), 95, 106, 107, 733
length, 221, 554
    curved, 71
    equal, 56
    of a parabola, 221
    of a path, 303
    of a wave, 582
    preservation of, 457
length infinitesimal, 555, 571
letter, 629, 747
Leucippus (fl. 440 B.C.), 109
level and ramp
    Fourier transform of, 661
level, ramp and, 658, 661
Levi-Civita epsilon, 514
    properties of, 516
Levi-Civita, Tullio (1873–1941), 514
library, programmer's, 432
liceity, 404
light ray, 524
likely stray, 697
limit, 97
limit of integration, 207, 664
line, 78, 430
    fitting a, 434
    parallel, 21
linear algebra, 329
linear combination, 215, 376, 409
linear dependence, 376, 409
linear expression, 25, 215, 313
linear independence, 376, 410
linear operator, 215
    unary, 537
linear quantity, 75
linear superposition, 258
linear system
    classification of, 410
    degenerate, 410
    exactly determined, 422
    nonoverdetermined, 431
    nonoverdetermined, general solution of, 432
    nonoverdetermined, particular solution of, 432
    overdetermined, 410, 433, 434
    taxonomy of, 410
    underdetermined, 410
linear transformation, 332
linearity, 215
    of a function, 215
    of an operator, 215, 537
    of the Fourier series, 595
ln, 133
loan, 286
localization, 647
locus, 171
logarithm, 49
    integrating a product of a power and, 308
    properties of, 50
    resemblance of to x^0, 136
logarithm, natural, 133
    and the antiderivative, 282
    compared to x^a, 134
    derivative of, 133, 150
    error of, 266
logarithmic cylindrical coordinates, 521

logarithmic derivative, 116, 132
of the gamma function, 723
of the natural exponential, 132
logic, 402, 731
reverse, 440
rigid, 734
symbolic, 402, 595
logical exercise, 57
logical flaw, 156
logical notation, 402, 595
logicofoundational program, 160
lone-element matrix, 345
long division, 38, 275
by $z - \alpha$, 169
procedure for, 41, 43
loop, 452
loop counter, 28
loop, surveyor's, 61
Lord Mayor's Show, 729
lore, 178
lower triangular matrix, 361
LU decomposition, 378
lurking, 297

Maclaurin series, 245
Maclaurin, Colin (1698–1746), 245
magnification, 476
magnitude, 65, 92, 137
of a vector, 445
of an eigenvalue, 476
of an integral, 292
preservation of, 457
unit, 476
main diagonal, 338, 485
majority, 452
majorization, 239, 263, 264
geometric, 265
maneuver, logical, 402
manipulation
of a Pythagorean expression, 303
of a Pythagorean nonradical, 306
of a Pythagorean radical, 303
of the derivative, 117
mantissa, 386, 432
mapping, 62
marking
permutor, 460
marking quasielementary, 460
Mascheroni, Lorenzo (1750–1800), 727
mason, 172
mass density, 219
matching coefficients, 42
mathematical formalism, 7
mathematical Platonism, 7
mathematician
applied, 2
applied, chief interest of, 8
professional, 3, 156, 302
mathematics
applied, 2, 231, 240
applied, foundations of, 329
concepts of, 161
professional or pure, 3, 156, 231, 751
matrix, 329, 330
addition of, 333
arithmetic of, 333
associativity of the multiplication of, 333
basic operations of, 332
basic use of, 332
broad, 410, 431
column of, 387, 425, 453
commutation of, 350
condition of, 476
degenerate, 409, 421
diagonalizable, 488
dimension-limited, 340
dimensionality of, 414
eigenvalue, 473
exact, 401
form of, 338
full-rank, 409
general identity, 341
Hermitian, 487
identity, 341, 344
identity, impossibility to promote of, 403
ill-conditioned, 469, 476
inversion of, 350, 417, 418, 467
inversion properties of, 354

large, 432, 468
lone-element, 345
main diagonal of, 338, 485
motivation for, 329, 332
multiplication of, 333, 338
multiplication of by a scalar, 333
noncommutivity of the multiplication of, 333
nondiagonalizable, 485
nondiagonalizable versus singular, 475
null, 340
null column of, 453
orthogonally complementary, 449
padding of with zeros, 338
parallel unit triangular, properties of, 369
perpendicular, 449
projection, 493
provenance of, 332
raised to a complex power, 474
rank of, 407
rank-r inverse of, 354
real, 436
rectangular, 421
row of, 387
scalar, 341
self-adjoint, 487
singular, 421, 476
singular-value, 491
sparse, 342
square, 338, 410, 412, 418
square, coïncident properties of, 468
tall, 410, 412
triangular, construction of, 363
truncating, 345
unit lower triangular, 361
unit parallel triangular, 365
unit triangular, 361
unit triangular, partial, 370
unit upper triangular, 361
unitary, 456
matrix operator, 338, 537
matrix rudiments, 329, 375
matrix vector, 330, 495
maximum, 120
Maxwell distribution, 704
Maxwell, James Clerk (1831–1879), 704
mean, 175, 681
arithmetic, 176
geometric, 176
harmonic, 176
inference of, 691
of a waveform, 593
mean square, 599
mean-value theorem, 12
means, justified by the end, 402
measure, unit of, 76, 580, 745
measurement
experimental, 662
mechanical bearing, 700
mechanical engineer, 612, 631
Melissus (fl. 440 B.C.), 109
membership, 29, 52
memory, computer, 432, 451
mental energy, 57
mental image of the infinitesimal, 107
meromorphic function, 249, 274, 723
metaduality, 626
method, 11
metric, 437
metric coefficient, 554
mile per hour, 77
mind's eye, 110
minimum, 120
Minkowski inequality, 448
Minkowski, Hermann (1864–1909), 448
minorization, 264
minute, 580
mirror, 58, 150
parabolic, 524
missile, 702
mnemonic, 501
mode of reason, 12
model, 4, 156, 506, 554
modular arithmetic, 191
modulo notation, 162
modulus, 65
Moivre, Abraham de (1667–1754), 92, 141
Moore, E. H. (1862–1932), 434

Moore-Penrose pseudoinverse, 434, 455
motion
  about a circle, 138
  perpendicular to a radial arm, 138
motivation, 184, 187, 422
motive
  to posit the normal distribution, 689
mountain road, 430
mountain, barren, 110
Muller, Mervin E. (1928–2018), 704
multiple
  least common, 169
multiple pole, 63, 294, 300
  enclosing, 259
multiple-valued function, 246, 247
multiplication, 28, 66, 92
  index of, 28
  of a vector, 500
  of matrices, 333, 338
  repeated, 137
multiplicative inversion, 22
multiplier, 329
multitarget addition operator, 359
  downward, 360
  leftward, 361
  rightward, 361
  upward, 360
multivariate Newton-Raphson iteration, 443

naïveté, 161, 709, 732
Napoleon, 429
natural exponential, 129
  compared to x^a, 134
  complex, 129
  derivative of, 130, 148
  error of, 267
  existence of, 129
  Fourier transform of, 642
  logarithmic derivative of, 132
  real, 129
natural exponential family of functions, 143
natural language, 729
natural logarithm, 133
  and the antiderivative, 282
  compared to x^a, 134
  derivative of, 133, 150
  error of, 266
  of a complex number, 140
natural logarithmic family of functions, 143
nature
  complex number in, 150
neighborhood, 245
nesting, 160, 199, 452
Newton, Sir Isaac (1642–1727), 95, 105, 124, 443, 733
Newton-Raphson iteration, 124, 313, 709
  multivariate, 443
Noë, Alva (1964–), 11
noise, 636
nonanalytic function, 69
nonanalytic point, 246, 257
nonanalytic pulse, 583
  dual transform of, 625
  properties of, 585
  support of, 585
  unit sum of series of, 585
nonanalyticity, 585
nonassociativity
  of the cross product, 504
noncommutivity
  of matrix multiplication, 333
  of the cross product, 503
  of unary linear operators, 537
nonconvergent series, 270
nondiagonalizable matrix, 485
  versus a singular matrix, 475
nonelementary function, 711
noninvertibility, 63, 420
nonlinearity, 429
nonnegative definiteness, 439, 473
nonoverdetermined linear system, 431
  general solution of, 432
  particular solution of, 432
nonradical, Pythagorean, 306
nonrectangular notation, 556
nonrepeating waveform, 615
nonstandard notation, 108

nonuniform sample, 661
normal distribution, 685, 687
  convergence toward, 698
  cumulative distribution function of, 688, 705
  motive to posit, 689
  proof of, 686
  quantile of, 709
normal unit vector, 507
normal vector or line, 224
normalization, 449, 453, 476
north, 71, 429
notation, 11
  for the vector, concise, 510
  for the vector, nonrectangular, 556
  logical, 402, 595
  nonstandard, 108
  of the operator, 537
  of the vector, 495
null column, 453
null matrix, 340
null vector, 376
null vector identity, 551
number, 64
  complex, 64, 92
  complex, actuality of, 150
  exact, 401
  imaginary, 64
  irrational, 159
  rational, 159
  rational, continued-fraction representation of, 180
  real, 64, 497
  very large or very small, 96
number theory, 153
  technique required for the physical applicationist to address, 191
numerator, 38, 293

observation, 662, 692
Observatory, Old Royal, 82
obviousness, 162
Ockham's razor
  abuse of, 151
Ockham, William of (c. 1287–1347), 151
octal, 159
odd function, 272
  Fourier transform of, 639
odd inverse square
  sum of, 602
odd square
  inverse, sum of, 602
oddness, 354
off-diagonal entries, 346
offset, 578
Old Royal Observatory, 82
one, 21, 71, 341
ontology, 8, 13
operator, 213, 535, 537
  + and − as, 214
  downward multitarget addition, 360
  elementary, 346
  general interchange, 357
  general scaling, 358
  leftward multitarget addition, 361
  linear, 215, 537
  multitarget addition, 359
  nonlinear, 215
  quasielementary, 356
  rightward multitarget addition, 361
  truncation, 345
  unary, 537
  unary linear, 537
  unresolved, 538
  upward multitarget addition, 360
  using a variable up, 213
operator notation, 537
optimality, 437
order, 25
  residual, 40
ordinary differential equation, 108
orientation, 78
origin, 71, 77
orthogonal basis, 505
  constant, 559
  variable, 506
orthogonal complement, 449
  by Gram-Schmidt, 455
  Gram-Schmidt formula for, 456
orthogonal vector, 445, 501

orthogonalization, 449, 454
orthonormal rows and columns, 457
orthonormal vectors, 89
orthonormalization, 417, 449
orthonormalizing decomposition, 451
inverting a matrix by, 458
oscillation, 611
outer edge, 547
outer surface, 545
overdetermined linear system, 410, 433, 434
overlap, 654, 663
overshot, 611

padding a matrix with zeros, 338
Palais, Bob, 224
parabola, 523
area within, 220
length of, 221
parabolic antenna, 524
parabolic arc, 523
parabolic coordinate grid, 526
parabolic coordinates, 522, 570
in two dimensions, 525
isotropy of, 528
properties of, 528
parabolic cross-section, 524
parabolic cylinder, 529
parabolic cylindrical coordinates, 529
parabolic mirror, 524
parabolic track, 525
paraboloid, 531
paraboloidal coordinates, 529
parallel addition, 172, 314
parallel lines, 21
parallel sides, 56
parallel subtraction, 175
parallel unit triangular matrix, 365
properties of, 369
parallelogram, 56
parameter, 191, 578
formal, 108
parity, 354, 459, 503
and the Levi-Civita epsilon, 514
Parmenides (born c. 515 B.C.), 109
Parseval's equality, 599
Parseval's principle, 295, 576
application of, 590
Parseval's theorem, 636
Parseval, Marc-Antoine (1755–1836), 295, 576, 599, 636
partial derivative, 108, 535
partial differential equation, 108
partial sum, 262, 707
partial unit triangular matrix, 370
partial-fraction expansion, 293
partial d, 108
particle, 704
particular solution, 431, 432
Pascal's triangle, 101
neighbors in, 99
Pascal, Blaise (1623–1662), 101
patch, 571
path integration, 229
path length, 303
pattern
derivative product, 120
filling of, 664
payment rate, 286
PDF (probability density function), 679
peasant levy, 762
pedantry, 3
pencil, 7, 395, 426, 705
Penrose, Roger (1931–), 434
period, 575
permutation, 98
permutation matrix, 357
permutor, 357, 459
marking, 460
perpendicular, 54
perpendicular matrix, 449
perpendicular unit vector, 507
perspective, Courant-Hilbert-Shilov, 5
persuasion, 167
perturbation, 294, 486
Pfufnik, Gorbag J., 249, 694
phase, 65, 137
phase factor
spatiotemporal, 669
phasor, 5

philology, 165
philosophy, 3
    Greek, 732
physical insight, 578
physical unit, 76, 578, 745
physical world, 495
physical-intuitional methods, 732
physicist, 5, 161, 669
pilot, 430
Pinter, Charles C. (1938–), 5
pitch, 498
pivot, 385
    small, 432
plain inverse, sum of, 602
Planck, Max (1858–1947), 46
plane, 78, 430
    projection onto, 519
Plato (428–348 B.C.), 10, 109
Platonism, 7
plausibility, 164
plausible assumption, 154
poem, 575
point, 78, 89, 430
    in vector notation, 77
Poisson distribution, 700
Poisson's ramp, 600
Poisson, Siméon Denis (1781–1840), 600, 700
pole, 63, 122, 246, 248, 257, 275
    circle of, 295
    double, 294, 300
    multiple, 63, 294, 300
    multiple, enclosing a, 259
    of a trigonometric function, 273
    of the gamma function, 722
    proper or distinct, 297
    repeated, 294, 300
    separation of, 294
    shadow, 297
polygon, 52
polynomial, 37, 169
    characteristic, 470, 485, 487
    having at least one root, 171
    Legendre, 727
    of order N having N roots, 170
position, 689
position vector, 509
positive definiteness, 438, 473
potential
    electric, 151
potential energy, 108
potentiometer, 430
power, 31, 75
    complex, 104
    electromagnetic, 504
    fractional, 33
    integral, 32
    notation for, 32
    of a power, 34
    of a product, 34
    properties of, 31
    real, 33
power series, 37, 69
    bounds on, 262
    common quotients of, 45
    derivative of, 106
    division of, 38
    division of by matching coefficients, 42
    extending the technique of, 46
    multiplication of, 38
    shifting the expansion point of, 241
    with negative powers, 37
power-related inequality, 36
premise, 11, 12
    implied, unstated, 12
pressure, 533
primality
    determination of, 158
primary frequency, 576
prime factor, 167
prime factorization, 154
prime mark ($'$), 78, 511
prime number, 153
    infinite supply of, 153
    relative, 162, 180, 201, 326
primed coordinate, 511
primitive guise, 575
probability, 677, 679
    a priori, 678

definitions pertaining to, 679
empirical, 678
that both of two independent events will occur, 680
probability density function, 679
flattening of, 684
of a sum of random variables, 680
processor, computer, 386
product, 28
determinant of, 466
dot or inner, 444
Fourier transform of, 631
of a vector and a scalar, 500
of determinants, 466
of vectors, 500
product rule, derivative, 118, 284, 373
of the contour, 552
of the vector, 548
pattern of, 120
productivity, 175
professional mathematician, 156, 302
professional mathematics, 3, 156, 231, 751
professor, 733, 736
profundity, 109
programming, 731
progression of coordinates, cyclic, 503
projectile, 523
projection
onto a plane, 519
projector, 493
prolixity, 510
promotion, 407
proof, 1
by contradiction, 154, 159, 403
by induction, 67, 238
by sketch, 4, 52, 557, 605
necessity of, 56
propagation speed, 582
propagation vector, 669
proper pole, 297
proper sequence, 355
proportion, 54, 182
proportional rate, 116
proportionality, 55
proposition, 11, 192
proving backward, 176
proviso, 424
prudence, 705
pseudoinverse, 434, 455
pseudorandom number, 699
pulse, 583, 615
arbitrary, 621
basic nonanalytic, 583
basic nonanalytic, dual transform of, 625
basic nonanalytic, properties of, 585
basic nonanalytic, support of, 585
basic nonanalytic, unit sum of series of, 585
collated, 654
contrasted against a pulse train, 644
Gaussian, 587, 644, 687
Gaussian, properties of, 587
Gaussian, to implement the Dirac delta by, 647
irregular triangular, 656
of unit area, 583
overlapping, 654
raised cosine-rolloff, 585, 588
raised-cosine, 583
raised-cosine, irregular, 663
right-triangular, 659
rolloff, 585
rolloff, properties of, 587
sharp, 583
square, 583
square, discontinuity in, 585
square, support of, 585
symmetrical, 583
time-limited, 654
trapezoidal, 585
triangular, 583
unit, 583
useful, 583
pulse train, 592, 615
contrasted against a single pulse, 644
Fourier coefficients of, 592
Fourier transform of, 643
rectangular, 592

pure mathematics, 3, 156, 231, 751
put-up job, 9
pyramid
  volume of, 224
Pythagoras (c. 580–c. 500 B.C.), 1, 54
Pythagorean expression, 303
Pythagorean nonradical, 306
Pythagorean radical, 303
Pythagorean theorem, 1, 54
  and the hyperbolic functions, 142
  and the sine and cosine functions, 73
  in three dimensions, 55

QR decomposition, 451
quadrant, 71
quadratic conjugation, 27
quadratic expression, 25, 172, 313, 316
quadratic formula, 26
quadratic repetition, 188
quadratic root, 184
quadratics, 25
quadrature, 639
quantile, 679
  of the normal distribution, 709
  use of to convert between distributions, 704
quantity
  exact, 401
  imprecise, 401
  inexact, 401
  linear, 75
  squared, 75
quartic expression, 25, 172, 313, 321
  resolvent cubic of, 323
  roots of, 324
quartic formula, 324
quasielementary operator, 356
  addition, 359
  interchange, 357
  marking, 460
  row-addition, 360
  scaling, 358
question
  tailoring, 109
quintic expression, 25, 172, 325
quiz, 508, 514
quotient, 38, 42, 293
  integral, 162

radial arm, 138
radian, 71, 82, 580
radical, Pythagorean, 303
radius, 71
raised cosine-rolloff pulse, 585, 588
  convergence related to, 619
  Fourier transform of, 619
  vanishing denominator in the Fourier transform of, 621
raised-cosine pulse, 583
  dual transform of, 625
  Fourier transform of, 619
  irregular, 663
ramp
  Poisson's, 600
ramp and level
  Fourier transform of, 658, 661
ramp, level and, 661
ramping property, 664
random variable, 679
  scaling of, 684
  sum of, 682
  transformation of, 684
random walk, 696
  consequences of, 698
range, 63, 425
range contour, 247
rank, 401, 407
  and independent rows, 392
  column, 410, 412, 423, 438
  full, 409
  impossibility of to promote, 403
  infinite, 410
  maximum, 392
  row, 410, 449
  uniqueness of, 407
rank-n determinant, 460
rank-r inverse, 354, 418
Raphson, Joseph (1648–1715), 124, 443
rate, 95

absolute, 116
of interest, 116
proportional, 116
relative, 116
ratio, 33, 159, 201, 293
construction of from the continued-fraction representation, 201
fully reduced, 159, 162
golden, 182
irreducible, 159, 162, 202
of gears, 581
rational function, 293
derivatives of, 300
integral of, 298
rational number, 159, 160
continued-fraction representation of, 180
rational root, 326
ray, 524
Rayleigh distribution, 702
Rayleigh, John Strutt, 3rd baron (1842–1919), 702
real argument, 634
real coordinate, 497
real exponential, 129
real number, 64, 160, 161, 497
approximation of as a ratio of integers, 33, 202
real part, 66
of the Fourier transform, 623
real-estate agent, 698
reason
circular, 165
mode of, 12
reciprocal, 22, 421
reciprocal gamma function, 723
reciprocal pair, 420
rectangle, 21
cutting of with scissors, 182
splitting of down the diagonal, 51
rectangular coordinates, 71, 89, 495
rectangular matrix, 421
rectangular pulse train, 592
Fourier coefficients of, 592
reduction, 159, 164
reference vector, 538
referential infinitesimal, 114
reflection, 54, 55
of the gamma function, 719
Reginald of Piperno, Father (c. 1230–c. 1290), 732
register, computer's floating-point, 432, 601
regular part, 258, 275
relative prime, 162, 180, 201
relative primeness, 326
relative rate, 116
remainder, 38, 162
after division by $z - \alpha$, 169
zero, 169
remarks, concluding, 729
Renaissance, 14
reorientation, 498
invariance under, 501, 502, 535, 550, 551
repeated eigensolution, 486
repeated eigenvalue, 473, 475, 486
repeated pole, 294, 300
repeating waveform, 575, 597
repetition, 652
of a sequence of bits, 180
quadratic, in the continued-fraction representation, 188
unseemly, 514
representation
by bit sequence, 180
by continued fraction, 178
residual, 265, 433, 705
minimizing the, 439
squared norm of, 434
residual order, 40
residue, 258, 293
of the gamma function, 722
residue theorem, Cauchy's impressed, 752
resolvent cubic, 323
restriction, 164
result
as unaltered by iteration, 182
retail establishment, 700
reversal of the independent variable, 283

reverse hypothesis, 197
reverse logic, 440
reversibility, 378, 407
revolution, 71, 579
Riemann, Georg Friedrich Bernhard (1826–1866), 8, 249, 597
right triangle, 51, 54, 71
right-hand rule, 78, 503, 506, 531
right-triangular pulse
  Fourier transform of, 659
rightward multitarget addition operator, 361
rigor, 3, 216, 240
  faux, 99
  to forestall error, 13
ringing, 611
rise, 71
road
  mountain, 430
  winding, 505
Robinson, Abraham (1918–1974), 111, 112
roll, 498
rolloff, 585
rolloff parameter, 585
rolloff pulse, 585
  properties of, 587
Roman alphabet, 747
roof, 534
root, 25, 33, 63, 122, 169, 313, 737
  double, 320
  finding of numerically, 124, 443
  guess at, 325
  rational, 326
  superfluous, 317
  triple, 320
root extraction
  from a cubic polynomial, 317
  from a quadratic polynomial, 26
  from a quartic polynomial, 324
root-length, 522
rotation, 54, 55, 78, 495
  angle of, 80
  Euler, 500
  Tait-Bryan or Cardan, 498
rounding, 178
rounding error, 432, 705
row, 330, 387
  addition of, 463
  null, 462
  null, appending a, 421
  orthonormal, 457
  scaled and repeated, 462
row operator, 335
row rank
  full, 410
row vector, 444
row-addition quasielementary, 360
Royal Observatory, Old, 82
RPM, 579
rudder, 430
rudeness, 165
rudiments, 329, 375
rugby, 113
run, 71

sales, 698
sample, 662, 690
  nonuniform, 661
sample statistic, 694
sampling, 621
sampling device, 622
sampling window, 622
Sands, Matthew (1919–2014), 696
SAS (side-angle-side), 58
satellite dish antenna, 524
Saturday, 434
scalar, 75, 330
  complex, 77
  condition of, 477
scalar field, 533, 554
  directional derivative of, 538
  gradient of, 538
scalar matrix, 341
scalar multiplication
  of a vector, 500
scale, 652
scaling, 54, 55, 461, 684
  of the independent variable, 284
scaling operator

elementary, 347
general, 358
scaling property of the Fourier transform, 629
scaling quasielementary, 358
scare quotes, 161
schematic, 395, 426
Schmidt, Erhard (1876–1959), 449
schools of mathematical thought, foundational, 11
Schur decomposition, 479
Schur, Issai (1875–1941), 362, 479
Schwarz inequality, 446
Schwarz, Hermann (1843–1921), 446
scissors, 182
screw, 78
screwdriver, 731
sea
wavy surface of, 507
second, 580
second derivative, 114, 121
secondary circle, 508
secondary cylindrical basis, 508
selection from among wooden blocks, 98
self-adjoint matrix, 487
semiconvergent series, 705, 708
semiperimeter, 59
separation of poles, 294
sequence
of integers, 178
product of, 28
proper or improper, 355
strictly decreasing, 163
sum of, 28
serial addition, 172
series, 28
arithmetic, 30
asymptotic, 705, 708
convergence of, 92
divergent, 270
Fourier, 575, 589
geometric, 45, 265
geometric, variations on, 46
harmonic, 265
multiplication order of, 29
notation for, 28
semiconvergent, 705, 708
Taylor, 235, 244
truncation of, 262, 707, 708
series addition, 172
set, 3, 29, 52, 160
set notation, 29, 52
set theory, 11, 731
shadow pole, 297
shape
area of, 223
sharp pulse, 583
shelf, 659
shift
in frequency, 626
shift operator, 372, 424
shifting an eigenvalue, 472
shifting an expansion point, 241
Shilov, Georgi E. (1917–1975), 5, 751
shortcut to the axis, 605
side
parallel, 56
side-angle-side (SAS), 58
side-side-angle (SSA), 59
side-side-side (SSS), 58
sifting property, 231, 337, 648, 652
sign
alternating, 262, 605
signal
discretely sampled, 585
signaling, 636
Silverman, Richard A. (1926–), 99
similar triangles, 54
similarity, 55, 478
similarity transformation, 350, 378, 478
Simpson's rule, 210
simultaneous system of linear equations, 333, 422
sinc function, 603
sine, 71, 504
approximation of to first order, 262
derivative of, 146, 148
Fourier transform of, 643
in complex exponential form, 142
Laplace transform of, 665

law of, 86
sine integral, 604
evaluation of by complex contour, 608
final value of, 606
properties of, 606
Taylor series for, 604
sine-argument function, 603
derivative of, 604
Fourier transform of, 640
integral of, 604
properties of, 604
Taylor series for, 603
single-valued function, 246, 247
singular matrix, 421, 476
determinant of, 465
versus a nondiagonalizable matrix, 475
singular value, 491
singular-value decomposition, 491
singular-value matrix, 491
singularity, 63
essential, 63, 250, 277
sink, 540
sinusoid, 73, 589
Laplace transform of, 665
superposition of, 575
Sirius, 7
sixfold integral, 219
skepticism, 476
sketch, proof by, 4, 52, 557, 605
skip-induction, 190
sky, 534
slide rule, 705
slope, 71, 121
arbitrarily steep, 597
infinite, 621
slow convergence, 267
slow function, 134
smoothness, 585
soil, 4
soldier, 762
solenoidal field, 551
solid
surface area of, 225
volume of, 220
solution, 417
error in, 477
family of, 431
general, 432
guessing the form of, 286
of least-squares, 436
particular, 432
particular and homogeneous, 431
verification of, 666
sophomore, 731
sound, 47
source, 366, 540
source-free field, 551
south, 71, 429, 507
space, 78, 398, 424, 579, 639, 669
address of, 413
three-dimensional, 504
two-dimensional, 504
space and time, 219
space shot, 111
spare column, 391, 412
sparsity, 342
spatial Fourier transform, 669
spatial frequency, 582, 639, 669
spatiotemporal phase factor, 669
special case, 164
special function, 711
spectral density, 636
speed, 77, 505
of propagation, 582
sphere, 90, 496, 507
surface area of, 225
volume of, 227
spherical basis, 507
spherical coordinates, 89, 495, 554
spherical surface, 555
spiral inward toward bounds, 194
split form, 431
square, 82
even inverse, sum of, 602
inverse, sum of, 601, 602
odd inverse, sum of, 602
rotated, 1, 12
sum or difference of, 25

tilted, 1, 12
square matrix, 338, 412, 418
coïncident properties of, 468
degenerate, 421
square pulse, 583
discontinuity in, 585
divergence related to, 619
dual transform of, 625
Fourier transform of, 619
support of, 585
square root, 33, 64, 184
calculation of by Newton-Raphson, 126
square wave, 575, 592
approximation to, 575
Fourier coefficients of, 592
variant on, 593
square, completing the, 26, 644
squared quantity, 75
squared residual norm, 434
squares, least, 434
squaring, 318
SSA (side-side-angle), 59
SSS (side-side-side), 58
stake, surveyor's, 61
standard deviation, 681
inference of, 693
state space, 667
statistic, 681
inference of, 690
sample, 694
statistics, 677
steepest rate, 539
step in every direction, 576
step, irregular, 660
Stokes' theorem, 547
Stokes, Sir George Gabriel (1819–1903), 547
stone, 110
stray, 697
strictly triangular matrix, 362
strip, tapered, 225
Student, a statistician, 696
style, 14, 156, 233
suaveness, 684
subimposition, 3
subscript
indicating the components of a vector by, 511
substruction, 7
subtraction
parallel, 175
sum, 28
continuous limit of, 615
partial, 262, 707
weighted, 376
sum of even inverse squares, 602
sum of functions, 652
sum of inverse squares, 601, 602
sum of inverses, 602
sum of odd inverse squares, 602
summand, 216
summation, 28
as integration, 611
compared to integration, 264
convergence of, 92
index of, 28
summation convention, Einstein's, 513, 556
Sunday, 7
superfluous root, 317
superposition, 150, 258, 462, 652, 661
of complex exponentials, 575
of pulses, 622
of sinusoids, 575
superpositional property of the Fourier transform, 629
support
of a basic nonanalytic pulse, 585
of a rolloff pulse, 587
supposition, 167
surface, 218
closed, 540, 545
inner and outer, 545
orientation of, 507
spherical, 555
surface area, 225
surface element, 547, 571
surface infinitesimal, 233, 555, 571
surface integration, 219, 225, 571

closed, 227
surface normal, 507
surveying, 61
swapping of indices, 446
sweeping out a length, 554
swimming pool, 430
symbolic logic, 402, 595
symmetrical pulse, 583
symmetry, 272, 400, 446
appeal to, 78, 578
in the Fourier transform, 623
of the Fourier transform, 675
system
continuous or discrete, 731

tail
of a function, 661
omission of, 663
suppression of, 663
tailoring the question, 109
Tait, Peter Guthrie (1831–1901), 498
Tait-Bryan rotations, 498
tall matrix, 412
tangent, 71
compared against its argument, 606
derivative of, 148
in complex exponential form, 141
tangent line, 124, 130
tapered strip, 225
target, 366
Tartaglia, Niccolò Fontana (1499–1557), 313
tautology, 510
taxonomy of linear systems, 410
Taylor expansion, first-order, 104, 262
Taylor series, 235, 244
analog of, 575
converting a power series to, 241
for specific functions, 260
for the sine integral, 604
for the sine-argument function, 603
in $1/z$, 277
integration by, 310
multidimensional, 279
transposition of to a different expansion point, 245
Taylor, Brook (1685–1731), 235, 244
teacher, 731
technician, 430
technique
number-theoretical, 191
tension, electric, 75, 151
tergiversation, 729
term
cross-, 279
finite number of, 262, 707
Thales (fl. 585 B.C.), 109
theory, 422
third derivative, 114
three-dimensional geometrical vector, 495
three-dimensional space, 504, 529
thumb, 677
Thursday, 535
time, 579, 639, 669
inverse, 579
lack of, 13
time and space, 219
time domain, 617, 666
time-limited pulse, 654
time-limitedness, 585
toleration, 110
transfer function, 631
transform, 615, 651
Fourier, 615
Laplace, 663
transform domain, 617, 666
transform pair, 617, 641, 664, 666, 674
transformation
Box-Muller, 704
linear, 332
of a random variable, 684
variable of, 617
transit of Venus, 729
transpose, 336, 463
conjugate, 336
of a matrix inverse, 353
trapezoid rule, 210
trapezoidal pulse, 585
travel, 138

tree, 534, 737
trial, 158, 678
trial derivative, 307
trial division, 158, 165
triangle, 51, 52, 86
    altitude of, 59
    area of, 51, 59
    equilateral, 82
    right, 51, 54, 71
    semiperimeter of, 59
    similar, 54
triangle inequalities, 52
    complex, 92, 292
    complex vector, 447
    vector, 92, 447
triangular matrix, 362, 485
    construction of, 363
    parallel, properties of, 369
    partial, 370
    unit parallel, 365
triangular pulse, 583
    dual transform of, 625
    Fourier transform of, 619
    irregular, Fourier transform of, 656
    overlapping, 663
    right-, Fourier transform of, 659
trigonometric family of functions, 143
trigonometric Fourier series, 598
trigonometric function, 71
    derivative of, 148
    Fourier transform of, 643
    inverse, 71
    inverse, derivative of, 150
    of a double or half angle, 82
    of a sum or difference of angles, 80
    of an hour angle, 82
    poles of, 273
trigonometrics
    complex, 141
    hyperbolic, 142
    inverse complex, 143
trigonometry, 71
    properties of, 74, 88
triple integral, 219, 669
triple root, 320
triviality, 376
truncation, 262, 394, 707, 708
    of a continued fraction, 178
truncation operator, 345
truths of mathematics, the, 110
Tuesday, 535
tuning, 430
two-dimensional geometrical vector, 497
two-dimensional space, 504, 525
Tymoczko, Thomas (1943–1996), 8

U.S. male, 677
unary linear operator, 537
unary operator, 537
    unresolved, 538
unbalanced expression of convolution, 631, 651
uncertainty, 677
uncontinued term, 185
    omission of, 188
underdetermined linear system, 410
undergraduate, 731
uniform distribution, 699
uniqueness, 302, 421
    of matrix rank, 407
unit, 64, 71, 75
    financial, 76
    imaginary, 64, 137
    of measure, 76, 580, 745
    physical, 76, 578, 745
    real, 64
unit area, 583
unit basis vector, 75
    cylindrical, 89, 556
    derivative of, 557
    spherical, 89, 556
    variable, 89, 556
unit circle, 71
unit lower triangular matrix, 361
unit magnitude, 476
unit normal, 507
unit pulse, 583
unit step function, Heaviside, 230
    Fourier transform of, 642
unit triangular matrix, 361

construction of, 363
parallel, properties of, 369
partial, 370
unit upper triangular matrix, 361
unit vector, 75, 445, 556
normal or perpendicular, 507
unitary matrix, 456
determinant of, 466
unitary similarity, 478
unitary transformation, 478
United States, 731
unity, 21, 71, 75
university, 736
unknown coefficient, 286
unproved assertion, 55
unreasonable function, 246
unresolved operator, 538
unsupported proposition, 12
unsureness, logical, 156
up, 71, 430, 507
upper triangular matrix, 361
upstairs, 534
upward multitarget addition operator, 360
utility variable, 86

vacillating consensus, 112
vanishing denominator
in the Fourier transform of a raised cosine-rolloff pulse, 621
vanity, 10
variable, 47
assignment, 25
change of, 25
complex, 16, 114, 751
definition notation for, 25
dependent, 47, 106
independent, 47, 106
independent, lack of, 108
independent, multiple, 108
independent, reversal of, 283
independent, scaling of, 284
random, 679
utility, 86
variable independent infinitesimal, 210
variable of transformation, 617
variable $d\tau$, 210
vector, 75, 279, 330, 495
n-dimensional, 330, 495
addition of, 497
algebraic identities of, 519
angle between two, 445, 501
arbitrary, 376, 398
building of from basis vectors, 478
column, 403, 444
concise notation for, 510
derivative of, 533
derivative of, in cylindrical coordinates, 559
derivative of, in spherical coordinates, 563
derivative of, nonrectangular, 557
derivative product rule for, 548, 552
differential identities of, 547
dot or inner product of two, 444, 501
driving, 423, 425, 431
elementary, 345, 403, 444
elementary, left-over, 405
ersatz, 535
generalized, 279, 330
geometrical, 75
integer, 279
local, 509
magnitude of, 445
matrix, 330, 495
multiplication of, 500
nonnegative integer, 279
nonrectangular notation for, 556
normalization of, 449
notation for, 75
orientation of, 444
orthogonal, 445, 501
orthogonalization of, 449
orthonormal, 89
orthonormalization of, 449
point, 77
position, 509
projection of onto a plane, 519
reference, 538
replacement of, 398, 428
rotation of, 78

row, 444
row of, 330
scalar multiplication of, 500
second-order derivative of, 550
three-dimensional, 76
three-dimensional geometrical, 495
two-dimensional, 75
two-dimensional geometrical, 497
unit, 75, 445, 556
unit basis, 75
unit basis, cylindrical, 89, 556
unit basis, derivative of, 557
unit basis, spherical, 89, 556
unit basis, variable, 89, 556
zero or null, 376
vector algebra, 495
vector analysis, 495
vector calculus, 533
definitions and identities of, 549
integral forms of, 545
vector field, 533
components of, 556
curl of, 542
decomposition of, 556
directional derivative of, 538
divergence of, 540
vector infinitesimal, 571
vector notation, 495
vector space, 398, 424, 425
address of, 413, 428
vector, propagation, 669
velocity, 505, 533
local, 509
Venus, transit of, 729
verbal cue, 580
verification of a solution, 666
vertex, 224
vertical rise, 71
Vieta's parallel transform, 314
Vieta's substitution, 314
Vieta's transform, 314, 315
Vieta, Franciscus (François Viète, 1540–1603), 313, 314
visual dimension or axis, 247
visualization, geometrical, 377
voltage, 151
volume, 21, 218, 220, 555
enclosed by a surface, 540, 545
in a spherical geometry, 555
of a cone or pyramid, 224
volume element, 545
volume integration, 219
volumetric infinitesimal, 233, 555

walk, random, 696
consequences of, 698
wall, 652
warning, 165
warped d, 108
wave
complex, 150
propagating, 150
square, 575, 592
wave mechanics, 669
wave number, 582
waveform
approximation of, 576
continuous and repeating, 597
discontinuous, 597, 610
mean value of, 593
nonrepeating, 615
real, 598
repeating, 575
wavelength, 582
wavy sea, 507
weather forecast, 111
Wednesday, 535
week
days of, 535
weekday, 7
Weierstrass, Karl Wilhelm Theodor (1815–1897), 239, 733
weighted sum, 376
west, 71, 429
West Point, 762
Weyl, Hermann (1885–1955), 7, 11, 111
Wilbraham, Henry (1825–1883), 610
wind, 4, 505, 533
winding, 171
winding road, 505

window of sampling, 622
Wittgenstein, Ludwig (1889–1951), 8, 110
wooden block, 98
worker, 434
world, physical, 495

x86-class computer processor, 386

yaw, 498
yen, Japanese, 76

Zeno of Cition (fl. 250 B.C.), 109
Zeno of Elea (fl. 460 B.C.), 109
Zermelo, Ernst (1871–1953), 11
Zermelo-Fraenkel and Choice set theory (ZFC), 11
zero, 21, 63
  division by, 97
  floating-point, 433
  matrix, 340
  padding a matrix with, 338
  vector, 340
zero matrix, 340
zero vector, 340, 376
ZFC, 11